

JACOBI COMPLEXES ON THE RAN SPACE

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ABSTRACT. We study local theory of moduli schemes using the framework of the Ran space. With the help of the study of sheaves and complexes over the Ran space by Beilinson and Drinfeld in their theory of chiral algebras, we revisit Ran's works on the Jacobi complexes (the Chevalley complexes for sheaves of Lie algebras on the Ran space), the universal deformation rings of moduli problems, the higher Kodaira-Spencer maps, and construction of Hitchin-type flat connections. We give rigorous treatments in the algebraic setting, which seems to be new.

CONTENTS

0. Introduction	1
Notation	2
Acknowledgements	2
1. Ran space and sheaves on it	2
1.1. Ran space	3
1.2. Sheaves on Ran space	3
1.3. Convolution tensor product	4
2. Jacobi complex	5
2.1. Chevalley complex	5
2.2. Homotopy property of Chevalley complex	6
2.3. Jacobi complex and the universal deformation ring	7
2.4. Jacobi complex with coefficients and moduli module	8
3. Higher Kodaira-Spencer maps	9
3.1. The statement	9
3.2. Maurer-Cartan equation	9
3.3. Resolution	10
3.4. The construction of universal family	10
3.5. Connecting morphism	11
3.6. The construction of higher Kodaira-Spencer maps	12
4. The moduli space of G -bundles	12
4.1. Lie algebroid	13
4.2. Higher Kodaira-Spencer map for Lie algebroid	13
4.3. The G -bundle case	13
5. Flat connection on the Jacobi homology	14
5.1. BV algebras	14
5.2. Filtered BV algebras	15
5.3. Chevalley complex	15
5.4. Rigidity	15
5.5. Relative rigidity	16
5.6. Hitchin connection	17
References	18

0. INTRODUCTION

The purpose of this note is to understand the works [R93, R00, R06, R08] by Z. Ran on local geometry of moduli spaces. His study is based on sheaves of dg Lie algebras on what he called the very symmetric space, which is now called the Ran space.

Let us briefly explain the notion of Ran space. The motivation of its definition comes from deformation theory, as explained in [R93, R00], Study on local geometry of moduli space is equivalent to deformation theory of the data considered. Ran's idea is that the n -th order deformations of objects related to X should be controlled by some sheaves on the space parametrising n -tuples of points on X . The latter space is the (n -th filtration of) Ran space $\mathcal{R}(X)_n$.

Precisely speaking, for a given space or scheme X , the Ran space $\mathcal{R}(X)$ is the topological space of finite points in X . Thus a point in $\mathcal{R}(X)$ is a finite tuple $\{x_1, \dots, x_n\} \subset X$, where x_i 's may coincide. It has the natural filtration $\mathcal{R}(X)_1 \subset \mathcal{R}(X)_2 \subset \dots$ with $\mathcal{R}(X)_n$ the subspace of n -tuples of points.

The Ran space looks too simple at the first glance. Actually it is surprisingly strong as the results in Ran's works show. Let us explain a few of them. In [R00], the universal deformation ring is constructed in terms of what he called the Jacobi complex. The Jacobi complex is the Chevalley complex of the dg Lie algebra on $\mathcal{R}(X)$, as we will see later in §2. In [R06], a general framework to construct a flat connection related to moduli spaces is presented. It includes the Hitchin connection [H90] on the space of generalized theta functions.

The first purpose of this note is to understand Ran's work in the algebraic setting. Basically Ran worked over a Hausdorff space X . Since for a scheme X the Ran space $\mathcal{R}(X)$ is not a scheme (nor an ind-scheme), one should take care to treat sheaves and complexes on $\mathcal{R}(X)$.

Another remark is that large part of Ran's construction utilizes a resolution of sheaves on $\mathcal{R}(X)$, so that it is natural to restate his results in a derived or homotopy setting. This is our second purpose.

Fortunately, at present we have a reference of sheaves on $\mathcal{R}(X)$. In [BD04] Beilinson and Drinfeld built the theory of chiral algebras, and they utilized the category of sheaves (and \mathcal{D} -modules) on $\mathcal{R}(X)$. We will fully use their treatment.

Let us say a few words on chiral algebras. They are some Lie objects in a non-standard tensor category of \mathcal{D} -modules on a given curve. The theory of chiral algebras is almost (but not exactly) equivalent to that of vertex algebras, which encodes the local symmetry of conformal field theory. In [BD04] the Ran space is used in the equivalence of chiral algebras and factorization algebras. The latter is a sheaf on $\mathcal{R}(X)$ with additional conditions.

Let us explain the organization of this note. In §1 we explain the Ran space and sheaves on it following [BD04]. Although the theory of chiral algebras use \mathcal{D} -modules on $\mathcal{R}(X)$ extensively, our study requires only \mathcal{O} -modules so that our presentation is a restricted version of [BD04]. In §2 we give the definition of Jacobi complex after the preparation of general study of Chevalley complex of Lie algebra objects in arbitrary tensor category. As an application of Jacobi complex, we construct the universal deformation ring and the higher Kodaira-Spencer maps for deformation of schemes in §3. A modified version for the moduli space of G -bundles is given in §4. In §5 we construct flat connections on the homology of Jacobi complex. The Hitchin connection is the basic example of this construction.

As we will see in the main text, our study of local geometry of moduli spaces has many common features with [BD04]. This is not so surprising, since conformal vertex algebras are intimately connected to moduli problems on algebraic curves, as explained in [FB04, Chap. 16–17]. Beyond such a conceptual explanation, we expect a kind of functorial correspondence from our study to the theory of chiral algebras. For example, our construction of flat connections in §5 goes along almost the same line as the construction of flat connections on chiral homology in [BD04]. In the case of Hitchin connection, the expected correspondence should be the so-called Verlinde isomorphism. We will study this problem in future.

Notation. For a set S , $|S|$ denotes its cardinality.

For a category \mathcal{C} , the notation $A \in \mathcal{C}$ means that A is an object of \mathcal{C} . The word “tensor category” is used in the meaning of a category \mathcal{C} with multiplication $\otimes_{\mathcal{C}}$ which is symmetric with the commutator $s_{A,B} : A \otimes_{\mathcal{C}} B \rightarrow B \otimes_{\mathcal{C}} A$ in the sense of Mac Lane. The word “symmetric monoidal category” is used in the meaning of tensor category with unit. The phrase “dg” means “differential graded” as usual.

We will work on a fixed field k . The symbol \otimes denotes the standard tensor product in the category of k -vector spaces unless otherwise stated.

The grading of a dg k -vector space (namely a complex) C is denoted by the superscript like C^p , and the differential is assumed to be of degree $+1$. The n -shift of C is denoted by $C[n]$ in the meaning of $C[n]^p = C^{n+p}$.

We also use the language of operads freely. Let us name [LV12] for a reference out of plenty numbers of literature. Lie denotes the operad of Lie algebras.

For a sheaf F on a topological space, the notation $t \in F$ means that t is a local section of F . For a complex F^\bullet of sheaves, $\mathcal{H}^i(F^\bullet)$ denotes the i -th cohomology sheaf.

Finally, \mathfrak{S}_n denotes the n -th symmetric group.

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1. RAN SPACE AND SHEAVES ON IT

Following [R00, §2.1] and [BD04, §3.4, §4.2], we recall some basic notions on Ran space and sheaves on it.

1.1. Ran space. For a topological space X , denote by $\mathcal{R}(X)$ the *Ran space* which is the set of all non-empty finite subsets in X with the strongest topology such that the obvious map

$$r_I : X^I \longrightarrow \mathcal{R}(X)$$

is continuous for any finite index set I . The point of $\mathcal{R}(X)$ associated to a finite subset $S \subset X$ is denoted by $[S]$.

For $n \in \mathbb{Z}$, denote by $\mathcal{R}(X)_n$ the subspace of $\mathcal{R}(X)$ consisting of $[S]$ such that $|S| \leq n$. We have a projection map

$$r_n : X^n \longrightarrow \mathcal{R}(X)_n = X^n / \sim,$$

where $(x_i)_{i=1}^n \sim (x'_i)_{i=1}^n$ if and only if $\{x_i\} = \{x'_i\}$. The map r_n is nothing but r_I with $I = \{1, \dots, n\}$.

We have an increasing filtration

$$\mathcal{R}(X)_0 = \emptyset \subset \mathcal{R}(X)_1 = X \subset \mathcal{R}(X)_2 = \text{Sym}^2(X) \subset \mathcal{R}(X)_3 \subset \dots \subset \mathcal{R}(X)_\infty := \mathcal{R}(X).$$

Here $\text{Sym}^n(X) := X^n / \mathfrak{S}_n$ is the usual symmetric product.

Let us set $\mathcal{R}(X)_n^\circ := \mathcal{R}(X)_n \setminus \mathcal{R}(X)_{n-1}$. It coincides with the complement of the partial diagonals in $\text{Sym}^n(X)$. Thus

$$\mathcal{R}(X)_n^\circ = U^{(n)} / \mathfrak{S}_n, \quad U^{(n)} := X^n \setminus \cup (\text{partial diagonals}). \quad (1.1.1)$$

$\mathcal{R}(X)_n^\circ$ is nothing but the configuration space of n points in X .

For any surjection $\pi : J \twoheadrightarrow I$ denote by

$$\Delta^{(\pi)} \equiv \Delta^{(J/I)} : X^I \hookrightarrow X^J, \quad (x_i)_{i \in I} \mapsto (y_j := x_{\pi(j)})_{j \in J}. \quad (1.1.2)$$

the diagonal embedding. Then we have $r_J \Delta^{(J/I)} = r_I$, and $\mathcal{R}(X)$ is the inductive limit of the spaces X^I with respect to these embeddings $\Delta^{(J/I)}$.

$\mathcal{R}(X)$ is a commutative semigroup under the continuous map

$$u : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow \mathcal{R}(X), \quad ([S], [T]) \mapsto [S \cup T]. \quad (1.1.3)$$

It is the direct limit of $u_{m,n} : \mathcal{R}(X)_m \times \mathcal{R}(X)_n \rightarrow \mathcal{R}(X)_{m+n}$ given by the same operation. We have the relation

$$r_{m+n} = u_{m,n} \circ (r_m \times r_n). \quad (1.1.4)$$

We also have a continuous map

$$v_m^n : \mathcal{R}(\mathcal{R}(X)_m)_n \longrightarrow \mathcal{R}(X)_{mn}, \quad [\{[S_1], \dots, [S_n]\}] \mapsto [S_1 \cup \dots \cup S_n]. \quad (1.1.5)$$

1.2. Sheaves on Ran space. Hereafter let X be a scheme over a field k with finite cohomological dimension. Sheaves on schemes mean the ones in the étale topology.

Let us introduce some notations for the sheaves on X , which will be used throughout this note. Denote by $\mathcal{Sh}(X)$ the category of sheaves of k -vector spaces on X . An \mathcal{O} -module on X means a quasi-coherent sheaf of \mathcal{O}_X -modules over X . $\mathcal{M}_{\mathcal{O}}(X)$ denotes the category of \mathcal{O} -modules. $\mathcal{CM}_{\mathcal{O}}(X)$ denotes the dg category of complexes of \mathcal{O} -modules on X , and $\mathcal{DM}_{\mathcal{O}}(X)$ denotes the corresponding derived category, namely the localization of $\mathcal{CM}_{\mathcal{O}}(X)$ by quasi-isomorphisms. For a morphism $f : X \rightarrow Y$ of k -schemes, f^* and f_* denotes the usual pull-back and push-forward functors on \mathcal{O} -modules.

Let \mathcal{S} be the category of finite non-empty sets and surjections. Following [BD04, §3.4.1] we introduce the notion of sheaves on the Ran space $\mathcal{R}(X)$.

Definition 1.2.1. An \mathcal{O} -module on $\mathcal{R}(X)$ is a rule F assigning to each $I \in \mathcal{S}$ an \mathcal{O}_{X^I} -module F_I and to each $\pi : J \twoheadrightarrow I$ in \mathcal{S} an isomorphism

$$\nu_F^{(\pi)} : \Delta^{(\pi)*} F_J \xrightarrow{\sim} F_I$$

of \mathcal{O}_{X^I} -modules compatible with the composition of surjections, namely for any $\rho : K \twoheadrightarrow J$ and $\pi : J \twoheadrightarrow I$ we have

$$\nu_F^{(\pi)} \circ \Delta^{(\pi)*} (\nu_F^{(\rho)}) = \nu_F^{(\pi \circ \rho)},$$

and also have $\nu_F^{(\text{id})} = \text{id}_{F_I}$. Denote by $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$ the category of \mathcal{O} -modules on $\mathcal{R}(X)$.

We have a similar definition of a sheaf of k -vector spaces on $\mathcal{R}(X)$ by replacing $\Delta^{(\pi)*}$ with $(\Delta^{(\pi)})^{-1}$. Denote by $\mathcal{Sh}(\mathcal{R}(X))$ the corresponding category.

We want to consider a derived category for $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$, but since this category is only exact in the sense of Quillen and not abelian, we need to take some detour to handle complexes of sheaves on $\mathcal{R}(X)$. Following [BD04, §4.2], let us consider a larger category of sheaves living on an enlarged ‘space’ $X^{\mathcal{S}}$ above $\mathcal{R}(X)$.

Let $X^{\mathcal{S}}$ be the diagram of schemes on the opposite category \mathcal{S}° given by

$$X^{\mathcal{S}} : I \mapsto X^I, \quad (\pi : J \twoheadrightarrow I) \mapsto (\Delta^{(\pi)} : X^I \hookrightarrow X^J).$$

Here $\Delta^{(\pi)}$ is the diagonal map given in (1.1.2). Note that it is a closed embedding.

Definition 1.2.2. A $!-\mathcal{O}$ -module on $X^{\mathcal{S}}$ is a rule F assigning to each $I \in \mathcal{S}$ an \mathcal{O}_{X^I} -module F_I and to each $\pi : J \twoheadrightarrow I$ a morphism of \mathcal{O}_{X^J} -modules

$$\theta_F^{(\pi)} : F_J \longrightarrow \Delta_*^{(\pi)} F_I$$

compatible with the compositions of surjections. Denote by $\mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})$ the category of $!-\mathcal{O}$ -modules on $X^{\mathcal{S}}$.

$\mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})$ is an abelian k -linear category, and the corresponding dg category of complexes and the derived category are denoted by $\mathcal{CM}_{\mathcal{O}}(X^{\mathcal{S}})$ and $\mathcal{DM}_{\mathcal{O}}(X^{\mathcal{S}})$ respectively. Similarly we can define a $!$ -sheaf of k -vector spaces on $X^{\mathcal{S}}$. Denote by $\mathcal{Sh}(X^{\mathcal{S}})$ the corresponding category. It is abelian and k -linear, and we denote by $\mathcal{CSh}(X^{\mathcal{S}})$ and $\mathcal{DSh}(X^{\mathcal{S}})$ the associated categories.

Now we want to consider a subcategory of $\mathcal{CM}_{\mathcal{O}}(X^{\mathcal{S}})$ formed by complexes on $\mathcal{R}(X)$. Note that $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$ is naturally a full subcategory of $\mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})$ by adjunction.

Definition 1.2.3. A complex $F \in \mathcal{CM}_{\mathcal{O}}(X^{\mathcal{S}})$ is called *admissible* if for each $\pi : J \rightarrow I$ in \mathcal{S} the morphism $\theta_F^{(\pi)}$ yields a quasi-isomorphism $R\Delta^{(\pi)!}F_J \xrightarrow{\sim} F_I$. The category of admissible complexes is denoted by $\mathcal{CM}_{\mathcal{O}}(X^{\mathcal{S}})_{\text{adm}}$.

$\mathcal{CM}_{\mathcal{O}}(X^{\mathcal{S}})_{\text{adm}}$ is a full dg subcategory of $\mathcal{CM}_{\mathcal{O}}(X^{\mathcal{S}})$ closed under quasi-isomorphisms, and yields a full triangulated subcategory $\mathcal{DM}_{\mathcal{O}}(X^{\mathcal{S}})_{\text{adm}} \subset \mathcal{DM}_{\mathcal{O}}(X^{\mathcal{S}})$. We have a similar discussion for the category $\mathcal{CSh}(X^{\mathcal{S}})$ of complexes of $!$ -sheaves of k -vector spaces. Thus we have subcategories $\mathcal{CSh}(X^{\mathcal{S}})_{\text{adm}}$ and $\mathcal{DSh}(X^{\mathcal{S}})_{\text{adm}}$.

Now we can state

Definition. The *derived category* $\mathcal{DM}_{\mathcal{O}}(\mathcal{R}(X))$ of \mathcal{O} -modules (resp. the *derived category* $\mathcal{DSh}(\mathcal{R}(X))$ of sheaves of k -vector spaces) on $\mathcal{R}(X)$ is defined respectively to be

$$\mathcal{DM}_{\mathcal{O}}(\mathcal{R}(X)) := \mathcal{DM}_{\mathcal{O}}(X^{\mathcal{S}})_{\text{adm}}, \quad \mathcal{DSh}(\mathcal{R}(X)) := \mathcal{DSh}(X^{\mathcal{S}})_{\text{adm}}.$$

For $F \in \mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})$, we have an \mathcal{S}° -diagram of k -vector spaces $I \mapsto \Gamma(X^I, F_I)$. Then we can define

$$\Gamma(X^{\mathcal{S}}, F) := \varinjlim_I \Gamma(X^I, F_I). \quad (1.2.1)$$

By the discussion in [BD04, §4.2.2], we have the derived functor

$$R\Gamma(X^{\mathcal{S}}, -) : \mathcal{DM}_{\mathcal{O}}(X^{\mathcal{S}}) \rightarrow \mathcal{D}(k),$$

where $\mathcal{D}(k)$ is the derived category of k -vector spaces.

For $F \in \mathcal{DM}_{\mathcal{O}}(X^{\mathcal{S}})$, we denote

$$H^{\bullet}(X^{\mathcal{S}}, F) := H^{\bullet}R\Gamma(X^{\mathcal{S}}, F).$$

Remark 1.2.4. As explained in [BD04, §4.2.4], in the situation where X is a locally compact Hausdorff space, $R(X)$ is the inductive limit of the diagram $X^{\mathcal{S}}$ and $\mathcal{Sh}(\mathcal{R}(X))$ is an abelian category thus one can define $\mathcal{DSh}(\mathcal{R}(X))$ directly. In this case one has an equivalence $\mathcal{DSh}(X^{\mathcal{S}})_{\text{adm}} \simeq \mathcal{DSh}(\mathcal{R}(X))$.

The restriction of the functor $R\Gamma(X^{\mathcal{S}}, -)$ to $\mathcal{DM}_{\mathcal{O}}(\mathcal{R}(X))$ is denoted by $R\Gamma(\mathcal{R}(X), -)$.

1.3. Convolution tensor product. We turn to a tensor structure on the categories of sheaves on $\mathcal{R}(X)$ and on $X^{\mathcal{S}}$ following [BD04, §3.4.10]. As in the previous subsection, X is a k -scheme of finite cohomological dimension.

Definition 1.3.1. For $F, G \in \mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})$, we define the *convolution product* $F \otimes^* G \in \mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})$ by

$$(F \otimes^* G)_I := \oplus_{\pi: I \rightarrow \{1,2\}} F_{\pi^{-1}(1)} \boxtimes G_{\pi^{-1}(2)}.$$

The structure morphisms $\theta^{(\pi)}$ are defined naturally. The resulting tensor category is denoted by $\mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})^*$. It induces a tensor structure on the full subcategory $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$, and the resulting tensor category is denoted by $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))^*$.

Let us compare this tensor structure with the natural one on the category $\mathcal{M}_{\mathcal{O}}(X)$ of \mathcal{O}_X -modules, namely the tensor product $\otimes_{\mathcal{O}_X}$. Denote by $\mathcal{M}_{\mathcal{O}}(X)^*$ the corresponding tensor category.

We have a projection functor

$$\mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}}) \rightarrow \mathcal{M}_{\mathcal{O}}(X), \quad F \mapsto F_{\{1\}},$$

where $\mathcal{M}_{\mathcal{O}}(X)$ is the category of \mathcal{O}_X -modules. It has a left adjoint

$$\Delta_*^{(\mathcal{S})} : \mathcal{M}_{\mathcal{O}}(X) \rightarrow \mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}}) \quad (1.3.1)$$

given by

$$(\Delta_*^{(\mathcal{S})} M)_I := \Delta_*^{(I)} M, \quad \theta^{(\pi: J \rightarrow I)} := \text{id}_{\Delta_*^{(J)} M}.$$

Here $\Delta^{(I)} := \Delta^{(I \rightarrow \{1\})} : X \hookrightarrow X^I$ is the diagonal map. The functor $\Delta_*^{(\mathcal{S})}$ is fully faithful.

The functor $\Delta_*^{(\mathcal{S})}$ induces $\mathcal{CM}_{\mathcal{O}}(X) \rightarrow \mathcal{CM}_{\mathcal{O}}(X^{\mathcal{S}})$, and in fact the admissibility in Definition 1.2.3 is automatically satisfied. Thus we have

$$\Delta_*^{(\mathcal{S})} : \mathcal{CM}_{\mathcal{O}}(X) \rightarrow \mathcal{CM}_{\mathcal{O}}(\mathcal{R}(X)), \quad \mathcal{DM}_{\mathcal{O}}(X) \rightarrow \mathcal{DM}_{\mathcal{O}}(\mathcal{R}(X))$$

Now we immediately have

Lemma 1.3.2. $\Delta_*^{(\mathcal{S})}$ is compatible with the tensor product $\otimes_{\mathcal{O}_X}$ and the convolution product \otimes^* . Thus it yields a fully faithful embedding of tensor category

$$\Delta_*^{(\mathcal{S})} : \mathcal{M}_{\mathcal{O}}(X)^* \hookrightarrow \mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})^*, \quad M \mapsto F = (F_I := \Delta_*^{(I)} M, \theta_F^{(\pi: J \rightarrow I)} := \text{id}_{\Delta_*^{(J)} M}).$$

By [BD04, §4.2.5], if X is quasi-compact, then the functors $\Gamma(X^S, -)$ and $R\Gamma(X^S, -)$ are tensor functors with respect to \otimes^* . So are the functors $\Gamma(\mathcal{R}(X), -)$ and $R\Gamma(\mathcal{R}(X), -)$

Remark 1.3.3. Continuing Remark 1.2.4, if X is Hausdorff, then \otimes^* on $\mathcal{M}_0(\mathcal{R}(X))$ can be written as

$$F \otimes^* G := u_*(F \boxtimes G), \quad (1.3.2)$$

where $u : \mathcal{R}(X) \times \mathcal{R}(X) \rightarrow \mathcal{R}(X)$ is the commutative semigroup structure given in (1.1.3). This convolution product is used in [R00] for the construction of the Jacobi complex and its OS-structure.

2. JACOBI COMPLEX

Let k be a field of characteristic 0.

2.1. Chevalley complex. Let us recall the Chevalley complex of Lie algebra. Since we will study Lie algebra objects in various categories, let us spell out in a general form. So let \mathcal{M} be an abelian k -linear symmetric monoidal category with the tensor product $\otimes_{\mathcal{M}}$. We will omit the symmetrizer $M \otimes_{\mathcal{M}} N \xrightarrow{\sim} N \otimes_{\mathcal{M}} M$ in the following presentation.

Denote by \mathcal{CM} the dg category of complexes of objects in \mathcal{M} . For a complex $V = (V^\bullet, d) \in \mathcal{CM}$, denote by $T^n(V)$ the n -th tensor power

$$T^n(V) := V \otimes_{\mathcal{M}} \cdots \otimes_{\mathcal{M}} V.$$

The dg tensor category \mathcal{CM} has the commutator

$$R_{V,W} : V \otimes_{\mathcal{M}} W \xrightarrow{\sim} W \otimes_{\mathcal{M}} V, \quad v \otimes w \mapsto (-1)^{|v| \cdot |w|} w \otimes v. \quad (2.1.1)$$

Here $|v|$ denotes the grading of $v \in V$. Under this tensor structure the symmetric group \mathfrak{S}_n acts on $T^n(V)$. Let $\text{Sym}^n(V)$ and $\wedge^n(V)$ be the spaces of coinvariants and the co-anti-invariants with respect to this \mathfrak{S}_n -action. Thus we have canonical projections denoted as

$$\begin{aligned} T^n(V) &\longrightarrow \text{Sym}^n(V), & v_1 \otimes \cdots \otimes v_n &\longmapsto v_1 \cdots v_n, \\ T^n(V) &\longrightarrow \wedge^n(V), & v_1 \otimes \cdots \otimes v_n &\longmapsto v_1 \wedge \cdots \wedge v_n. \end{aligned}$$

If the tensor category \mathcal{M} has a unit $1_{\mathcal{M}}$, then there is a natural unital commutative dg \mathcal{O}_S -algebra structure on the direct sum

$$\text{Sym}(V) := \bigoplus_{n \geq 0} \text{Sym}^n(V), \quad \text{Sym}^0(V) := 1_{\mathcal{M}}.$$

It has also the coproduct $\Delta : \text{Sym}(V) \rightarrow \text{Sym}(V) \otimes_{\mathcal{M}} \text{Sym}(V)$ determined by

- (1) $\Delta(v) = v \otimes 1 + 1 \otimes v$ for $v \in \text{Sym}^1(V) = V$,
- (2) Δ is a morphism of dg algebras in \mathcal{M} , where on $\text{Sym}(V) \otimes_{\mathcal{M}} \text{Sym}(V)$ the algebra structure is defined using the commutator (2.1.1).

Then together with the canonical projection $\varepsilon : \text{Sym}(V) \rightarrow 1_{\mathcal{M}} = \text{Sym}^0(V)$ as the counit, $\text{Sym}(V)$ is a commutative and cocommutative dg Hopf algebra in \mathcal{M} . It has an increasing filtration

$$\text{Sym}(V)_0 \subset \text{Sym}(V)_1 \subset \cdots \subset \text{Sym}(V), \quad \text{Sym}(V)_n := \bigoplus_{i=0}^n \text{Sym}^i(V). \quad (2.1.2)$$

Remark 2.1.1. Note that this filtration respects the augmentation structure on $S := \text{Sym}^i(V)$. Namely, denoting the augmentation ideal by $S^+ := \ker(\varepsilon) = \bigoplus_{i \geq 1} \text{Sym}^i(V)$ and by $\pi : S \rightarrow S^+$ the projection, we have $S_{n-1} = \ker(S \rightarrow S^{\otimes n} \rightarrow (S^+)^{\otimes n})$, where the first arrow is the n -th composition $\Delta^{(n)}$ of the coproduct, and the second one is $\pi^{\otimes n}$.

Hereafter in this subsection we assume \mathcal{M} is monoidal and consider the cocommutative dg coalgebra

$$C'(V) := (\text{Sym}(V[1]), d', \Delta). \quad (2.1.3)$$

Recall that the shift $[n]$ of complexes yield the canonical isomorphisms

$$V[m] \otimes_{\mathcal{M}} W[n] \xrightarrow{\sim} (V \otimes_{\mathcal{M}} W)[m+n], \quad v \otimes w \mapsto (-1)^{pn} v \otimes w \quad (2.1.4)$$

with $v \in V^p$. The isomorphisms (2.1.4) induce $t_n : T^n(V[1]) \xrightarrow{\sim} T^n(V)[n]$ with $t_n \circ \sigma = \text{sgn}(\sigma) \sigma \circ t_n$ for any $\sigma \in \mathfrak{S}_n$. Thus we have a canonical isomorphism

$$\text{Sym}^n(V[1]) \xrightarrow{\sim} \wedge^n(V)[n], \quad v_1 \cdots v_n \mapsto (-1)^{\sum_i (n-i)p_i} v_1 \wedge \cdots \wedge v_n, \quad (2.1.5)$$

where p_i is given by $v_i \in V^{p_i}$.

Let $L = (L^\bullet, d_L, [\cdot, \cdot])$ be a dg Lie algebra in \mathcal{M} . Thus $(L^\bullet, d_L) \in \mathcal{CM}$ and the Lie bracket $[\cdot, \cdot] : L \otimes_{\mathcal{M}} L \rightarrow L$ is a graded morphism which should satisfy

- (1) the graded skew-symmetry $[x, y] = -(-1)^{|x| \cdot |y|} [y, x]$,
- (2) the graded Jacobi identity $[x, [y, z]] + (-1)^{|x|(|y|+|z|)} [y, [z, x]] + (-1)^{|z|(|x|+|y|)} [z, [x, y]] = 0$,
- (3) the graded Leibniz rule $d[x, y] = [dx, y] + (-1)^{|x|} [x, dy]$.

By the construction (2.1.3), for a dg Lie algebra L we have a cocommutative dg coalgebra

$$C'(L) = (C^\bullet(L), d', \Delta).$$

Let us rewrite the dg coalgebra structure under the isomorphism $C^\bullet(L) \simeq \oplus_{n \geq 0} \wedge^n(L)[n]$ in (2.1.5). Let $x_i \in L^{\alpha_i}$ for $i = 1, \dots, n$, and for $I = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ with $i_1 < \dots < i_p$ we set

$$\bar{I} := \{1, \dots, n\} \setminus I, \quad x_I := x_{i_1} \wedge \dots \wedge x_{i_n} \in \wedge^n(S).$$

We also set $x_\emptyset := 1 \in 1_{\mathcal{M}}$. Then

$$\Delta(x_1 \wedge \dots \wedge x_n) = \sum_I \text{sgn}(I; \alpha_1, \dots, \alpha_n) x_I \otimes x_{\bar{I}} \quad (2.1.6)$$

Here $\text{sgn}(I; \alpha_1, \dots, \alpha_n) \in \{\pm 1\}$ is determined by the equation

$$x_1 \wedge \dots \wedge x_n = \text{sgn}(I; \alpha_1, \dots, \alpha_n) x_I \wedge x_{\bar{I}}.$$

Setting $\wedge^n(L) := 0$ for $n < 0$, the grading structure $C^\bullet(L)$ is described by

$$C^n(L) = \oplus_{p+q=n} C^{p,q}, \quad C^{p,q} := (\wedge^{-p}(L))^q = \text{span} \{x_{i_1} \wedge \dots \wedge x_{i_{-p}} \mid x_{i_k} \in L^{\alpha_k}, \sum \alpha_k = q\}.$$

Finally we rewrite the differential d' . The restriction of $d' : C^{p+q}(L) \rightarrow C^{p+q+1}(L)$ to $C^{p,q} \subset C^{p+q}(L)$ is given by

$$d' : C^{p,q} \rightarrow C^{p,q+1}, \quad x_1 \wedge \dots \wedge x_{-p} \mapsto (-1)^p \sum_{1 \leq i \leq -p} x_1 \wedge \dots \wedge d_L x_i \wedge \dots \wedge x_{-p}.$$

Now we can modify this differential by

$$\begin{aligned} d &:= d' + d'', \quad d'' : C^{p,q} \rightarrow C^{p+1,q}, \\ d''(x_1 \wedge \dots \wedge x_{-p}) &:= \sum_{1 \leq i < j \leq -p} \text{sgn}(\{i, j\}; \alpha_1, \dots, \alpha_n) [x_i, x_j] \wedge x_{\overline{\{i, j\}}}. \end{aligned} \quad (2.1.7)$$

Then we have $d^2 = (d'')^2 = 0$. The complex $C(L) = (C^\bullet(L), d)$ is nothing but the Chevalley-Eilenberg complex of the Lie algebra L .

Definition. $C(L) = (C^\bullet(L), d, \Delta)$ is called the *Chevalley complex* of L . We always consider it equipped with the filtration

$$C(L)_n = \oplus_{i=0}^n \text{Sym } S^i(L[1]) \simeq \oplus_{i=0}^n \wedge^i(L)[i]. \quad (2.1.8)$$

The truncated complex

$$\bar{C}(L) := (\text{Sym}^{\geq 1}(L), d, \Delta)$$

is called the *reduced Chevalley complex* of L . It is equipped with the same filtration $\bar{C}(L)_\bullet$ as (2.1.8)

Note that to define the reduced Chevalley complex it is not necessary to require the category \mathcal{M} to have a unit.

Remark. In [HS97] $C(L)$ is called the Quillen standard complex following [Q69, Appendix B].

2.2. Homotopy property of Chevalley complex. We discuss the homotopy property of the Chevalley complex. As in the previous subsection, let \mathcal{M} be an abelian k -linear monoidal category. Denote by \mathcal{CM} the dg category of complexes in \mathcal{M} as before.

Let us recall the notion of filtered quasi-isomorphisms of complexes. A *filtered complex* in \mathcal{M} is a complex C in \mathcal{M} with an increasing filtration C_\bullet . Denote by \mathcal{CFM} the category of filtered complexes C in \mathcal{M} such that $C_n = 0$ for $n \ll 0$ and $C = \cup_i C_i$. Morphisms in \mathcal{CFM} are those of complexes respecting the filtrations. A morphism $f : C \rightarrow C'$ of filtered complexes is called a *filtered quasi-isomorphism* if the induced morphism $\text{gr}_i(f) : \text{gr}_i(C) \rightarrow \text{gr}_i(C')$ on the associated graded is a quasi-isomorphism for any i .

Hereafter we use

Definition 2.2.1. Denote by \mathcal{HoC} the homotopy category of a closed model category \mathcal{C} .

As is well-known, the dg category of complexes is a closed model category with weak equivalences being filtered quasi-isomorphisms and fibrations being surjective morphisms. The dg category of filtered complexes is also a closed model category with weak equivalences being filtered quasi-isomorphisms and fibrations those morphisms f such that $\text{gr}(f)$ is surjective.

Let us consider the functor $L \mapsto C(L)$ of associating the Chevalley complexes to dg Lie algebras L in \mathcal{M} . As explained in the previous subsection, $C(L)$ has a filtration so that $C(L) \in \mathcal{CFM}$.

Definition 2.2.2. Denote by $\mathcal{Lie}(\mathcal{CM})$ the category of dg Lie algebras L in \mathcal{M} such that

- (1) every component L^n of the complex L is flat. Namely, the functor $L^n \otimes_{\mathcal{M}} -$ is exact.
- (2) $\mathcal{H}^n(L) = 0$ for $n \gg 0$.

A morphism in $\mathcal{Lie}(\mathcal{CM})$ is that in $\mathcal{Lie}(\mathcal{CM})$ respecting the Lie brackets. A morphism f is called a quasi-isomorphism if it is a quasi-isomorphism in \mathcal{CM} .

Now we have

Lemma. The functors $\mathcal{L}ie(\mathcal{CM}) \rightarrow \mathcal{CFM}$ given by $L \mapsto C(L)$ and $L \mapsto \overline{C}(L)$ sends quasi-isomorphism to filtered quasi-isomorphisms.

The proof is as in [HS97, §5.1.4 Lemma] so we omit it. Thus the functor $L \mapsto C(L)$ descends to the homotopy category and we obtain

Corollary. There are functors of homotopy categories

$$C, \overline{C} : \mathcal{HoL}ie(\mathcal{CM}) \longrightarrow \mathcal{HoCFM}, \quad L \longmapsto C(L), \overline{C}(L).$$

2.3. Jacobi complex and the universal deformation ring. In [R00] the Jacobi complex is defined as the Chevalley complex of a dg Lie algebra of sheaves on the Ran space $\mathcal{R}(X)$ assuming X to be a Hausdorff space. Here we introduce an analogue for the scheme setting.

Let X be a k -scheme. Recall the convolution product \otimes^* on the category $\mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})$ of $!-\mathcal{O}$ -modules in Definition 1.3.1. The resulting tensor category $\mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})^*$ is abelian and k -linear. It has a unit with $\mathcal{O} := (\mathcal{O}_{X'})$ with obvious $\theta(\pi)_{\mathcal{S}}$.

Now we can apply the argument in the previous §2.1 to the category $\mathcal{M} = \mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})^*$. A dg Lie \mathcal{O} -algebra on $X^{\mathcal{S}}$ is the complex of $!-\mathcal{O}$ -modules in $X^{\mathcal{S}}$ with the Lie algebra structure.

Definition 2.3.1. For a dg Lie \mathcal{O} -algebra L on $X^{\mathcal{S}}$, we have cocommutative dg coalgebras

$$C(L) = (\mathrm{Sym}(L[1]), d, \Delta), \quad \overline{C}(L) = (\mathrm{Sym}^{\geq 1}(L[1]), d, \Delta),$$

where the differential d is given by the formula (2.1.7) and the coproduct Δ in (2.1.6). We always consider them with the filtrations $C(L)_n := \bigoplus_{i=0}^n \mathrm{Sym}^i(L[1])$ and $\overline{C}(L)_n := \bigoplus_{i=1}^n \mathrm{Sym}^i(L[1])$. We call $C(L)$ the *Chevalley complex* and $\overline{C}(L)$ the *reduced Chevalley complex* of L .

We can also discuss the homotopy property as in the last paragraph of §5.3, Let us start with

Definition. Denote by $\mathcal{L}ie(X^{\mathcal{S}})$ the category of dg Lie \mathcal{O} -algebras on $X^{\mathcal{S}}$ consisting of objects $L = (L_I)$ such that

- (1) every component L_I^n of the complex L_I is $\mathcal{O}_{X'}$ -flat for any I ,
- (2) we have the vanishing of the the cohomology sheaf $\mathcal{H}^n(L_I) = 0$ for $n \gg 0$ and any I .

Now we have the next claim whose proof is the same as in [HS97, §5.1.4 Lemma].

Lemma. If f is a quasi-morphism in $\mathcal{L}ie(X^{\mathcal{S}})$, then the image $C(f)$ under the functor C is a filtered quasi-isomorphism.

As a corollary we have

Corollary 2.3.2. The correspondences $L \mapsto C(L)$ and $L \mapsto \overline{C}(L)$ yields the functors

$$C, \overline{C} : \mathcal{HoL}ie(X^{\mathcal{S}}) \longrightarrow \mathcal{HoCM}_{\mathcal{O}}(X^{\mathcal{S}}).$$

Now we turn to the definition of the Jacobi complex. Recall also the fully faithful embedding

$$\Delta_*^{(\mathcal{S})} : \mathcal{M}_{\mathcal{O}}(X)^* \hookrightarrow \mathcal{M}_{\mathcal{O}}(X^{\mathcal{S}})^*$$

given in Lemma 1.3.2. It naturally extends to the embedding $\Delta_*^{(\mathcal{S})} : \mathcal{CM}_{\mathcal{O}}(X) \hookrightarrow \mathcal{CM}_{\mathcal{O}}(X^{\mathcal{S}})^*$.

Let \mathfrak{g} be a dg Lie \mathcal{O}_X -algebra, namely a Lie algebra object in $\mathcal{CM}_{\mathcal{O}}(X)^*$. The image $\Delta_*^{(\mathcal{S})}(\mathfrak{g}) \in \mathcal{CM}_{\mathcal{O}}(X^{\mathcal{S}})^*$ is a Lie object. In order to ensure $\Delta_*^{(\mathcal{S})}(\mathfrak{g}) \in \mathcal{L}ie(X^{\mathcal{S}})^*$, we consider

Definition 2.3.3. Denote by $\mathcal{L}ie(X)$ the category of dg Lie \mathcal{O}_X -algebras \mathfrak{g} such that

- (1) every component \mathfrak{g}^n of the complex \mathfrak{g} is \mathcal{O}_X -flat,
- (2) $\mathcal{H}^n(\mathfrak{g}) = 0$ for $n \gg 0$,

and morphisms are those in $\mathcal{CM}_{\mathcal{O}}(X)$ respecting the Lie bracket.

If $\mathfrak{g} \in \mathcal{L}ie(X)$, then we obviously have $\Delta_*^{(\mathcal{S})}(\mathfrak{g}) \in \mathcal{L}ie(X^{\mathcal{S}})$. Now we have the main definition.

Definition 2.3.4. For $\mathfrak{g} \in \mathcal{L}ie(X)$, define the *Jacobi complex* $J(\mathfrak{g})$ to be the reduced Chevalley complex of $\Delta_*^{(\mathcal{S})}(\mathfrak{g})$.

$$J(\mathfrak{g}) := \overline{C}(\Delta_*^{(\mathcal{S})}(\mathfrak{g})) = (\mathrm{Sym}^{\geq 1}(\Delta_*^{(\mathcal{S})}(\mathfrak{g})[1]), d, \Delta).$$

Using the filtration on the reduced Chevalley complex, define

$$J_n(\mathfrak{g}) := \overline{C}(\Delta_*^{(\mathcal{S})}(\mathfrak{g}))_n.$$

Remark. Originally in [R00] the n -th term of the Jacobi complex of \mathfrak{g} is defined as the \mathfrak{S}_n -anti-invariant part of the sheaf $r_{n*}(\mathfrak{g}^{\boxtimes n})$ on $\mathcal{R}(X)$. Here $r_n : X^n \rightarrow \mathcal{R}(X)_n \subset \mathcal{R}(X)$ is the natural projection. In [R00] X is assumed to be Hausdorff so that this definition cannot be compared to ours strictly. However let us explain that these two definitions are essentially the same.

The relation (1.1.4) yields $r_n = v_1^n \circ (r_1 \times \cdots \times r_1)$, where v_1^n is given in (1.1.5). Then by the description (1.3.2) of \otimes^* for Hausdorff X we have $r_{n*}(\mathfrak{g}^{\boxtimes n}) = (r_{1*}\mathfrak{g}) \otimes^* \cdots \otimes^* (r_{1*}\mathfrak{g})$. In our case r_{1*} corresponds to $\Delta_*^{(S)}$, and taking the anti-invariant part is covered by considering $S(\mathfrak{g}[1]) \simeq \oplus_n \wedge^n(\mathfrak{g})[n]$. Thus the two definitions are basically the same one.

Note that we have the commutativity of the composition of functors $C \circ \Delta_*^{(S)} = \Delta_*^{(S)} \circ C$.

Let us briefly mention the homotopy property of Jacobi complex in a general setting. A quasi-isomorphism $\mathfrak{g} \rightarrow \mathfrak{h}$ in $\mathcal{L}ie(X)$ gives a quasi-isomorphism $\Delta_*^{(S)}(\mathfrak{g}) \rightarrow \Delta_*^{(S)}(\mathfrak{h})$. Then Corollary 2.3.2 gives

Lemma 2.3.5. The functors $\mathfrak{g} \mapsto J(\mathfrak{g})$ and $\mathfrak{g} \mapsto J_n(\mathfrak{g})$ from $\mathcal{L}ie(X)$ induce those from $\mathcal{H}o\mathcal{L}ie(X)$.

Recall the global section functor $\Gamma(X^S, -)$ in (1.2.1). The coproduct Δ on $J_n(\mathfrak{g})$ induces a ring structure on

$$R_n^u(\mathfrak{g}) := k \oplus (\Gamma(X^S, J_n(\mathfrak{g})))^*,$$

making $R_n^u(\mathfrak{g})$ an Artin local k -algebra of exponent n . Since the filtration and Δ is compatible in the sense of Remark 2.1.1, $(R_n^u(\mathfrak{g}))_{n \geq 1}$ form a direct system of Artin algebras, and it has the limit

$$R^u(\mathfrak{g}) := k \oplus (\Gamma(X^S, J(\mathfrak{g})))^* \simeq \varinjlim_n R_n^u(\mathfrak{g}).$$

Note that seen as an object of $\mathcal{CM}_{\mathcal{O}}(X^S)$, $J_n(\mathfrak{g})$ satisfies the admissible condition in Definition 1.2.3. so we may rewrite

$$R_n^u(\mathfrak{g}) = k \oplus (\Gamma(\mathcal{R}(X), J_n(\mathfrak{g})))^*.$$

Following [R00], we name

Definition. For $\mathfrak{g} \in \mathcal{H}o\mathcal{L}ie(X)$, we call the Artin algebra $R^u(\mathfrak{g})$ the *universal deformation algebra* of \mathfrak{g} . $R_n^u(\mathfrak{g})$ will be called the *n -th universal deformation algebra* of \mathfrak{g} .

2.4. Jacobi complex with coefficients and moduli module. Recall that for a Lie algebra \mathfrak{g} and a \mathfrak{g} -module V we have the Chevalley complex $C(\mathfrak{g}, V)$ with coefficients in V . Thus we can make a similar argument to obtain the Jacobi complex with coefficients in a module.

Let \mathcal{M} be an abelian k -linear tensor category as before, and $L \in \mathcal{CM}$ be a dg Lie algebra in \mathcal{M} . Let also M be a dg L -module in \mathcal{M} . Namely $M \in \mathcal{CM}$ with a \mathfrak{g} -action $\rho : \mathfrak{g} \otimes_{\mathcal{M}} V \rightarrow V$. As in the classical case, define the *reduced Chevalley complex of L with coefficients in V* to be

$$\overline{C}(L, M) := (\mathrm{Hom}_{\mathcal{M}}(\overline{C}(L), M), \partial).$$

The differential ∂ is given by

$$\partial(g) := \overline{\partial}(g) - (-1)^{|g|}gd$$

for $g \in \mathrm{Hom}_{\mathcal{M}}(\overline{C}(L), M)$. Here d is the differential of the complex $\overline{C}(L)$, and $\overline{\partial}(g)$ is given by the composition

$$\overline{C}(L) \xrightarrow{\Delta} \overline{C}(L) \otimes_{\mathcal{M}} \overline{C}(L) \xrightarrow{\pi \otimes g} L[1] \otimes_{\mathcal{M}} V \xrightarrow{\rho} V[1],$$

where the morphism $\pi : \overline{C}(L) \rightarrow L[1] = \mathrm{Sym}^1(L[1])$ is the canonical projection.

We want to apply this argument to the situation in the previous §2.3. So let X be a k -scheme of finite cohomological dimension, and $\mathfrak{g} \in \mathcal{L}ie(X)$. Also let V be a \mathfrak{g} -module in $\mathcal{CM}_{\mathcal{O}}(X)^*$, namely V is a complex of \mathcal{O}_X -modules with a \mathfrak{g} -action $\mathfrak{g} \otimes_{\mathcal{O}_X} V \rightarrow V$. We set $\mathcal{M} = \mathcal{M}_{\mathcal{O}}(X^S)^*$, $L = \Delta_*^{(S)}(\mathfrak{g})$ and $M = \Delta_*^{(S)}(V)$. Since $\Delta_*^{(S)}$ is a tensor functor by Lemma 1.3.2, M is an L -module. Thus we have the complex

$$J_n(\mathfrak{g}, V) := \overline{C}(\Delta_*^{(S)}(\mathfrak{g}), \Delta_*^{(S)}(V))_n.$$

It has a coproduct induced by Δ on the \mathfrak{g} -factor. It is natural to name

Definition 2.4.1. $J_n(\mathfrak{g}, V)$ is called the *n -th Jacobi complex with coefficients in V* .

The coproduct structure yields that the cohomology sheaf $\mathcal{H}^0(J_n(\mathfrak{g}, V))$ on X^S is a sheaf of $R_n^u(\mathfrak{g})$ -modules.

Let us define k -modules

$$M_n(\mathfrak{g}, V) := \Gamma(X^S, J_n(\mathfrak{g}, V)).$$

It is naturally an $R_n^u(\mathfrak{g})$ -module.

Definition. We call the $R_n^u(\mathfrak{g})$ -module $M_n(\mathfrak{g}, V)$ the *n -th moduli module* of V .

3. HIGHER KODAIRA-SPENCER MAPS

3.1. The statement. Let k be a field of characteristic 0, and X be a smooth scheme over k which is assumed to be separated and quasi-compact. We study the Jacobi complex of the tangent sheaf Θ_X , namely

$$J := J(\Theta_X) \supset J_n := J_n(\Theta_X)$$

given by Definition 2.3.4. Let us denote the corresponding universal deformation algebra by

$$R_n^u := R_n^u(\Theta_X) = k \oplus (\Gamma(\mathcal{R}(X), J_n))^*,$$

which is an Artin local k -algebra of exponent n .

In this section we explain the following result of [R00].

Theorem 3.1.1. Assume $H^0(X, \Theta_X) = 0$.

- (1) For each $n \in \mathbb{Z}_{\geq 1}$, there is a flat deformation \mathcal{X}_n^u of X over $\text{Spec}(R_n^u)$. The data $\{(\mathcal{X}_n^u, R_n^u)\}_{n \geq 1}$ form a direct system with the limit (\mathcal{X}^u, R^u) , which is a flat formal deformation \mathcal{X}^u over $\text{Spf}(R^u)$.
- (2) \mathcal{X}_n^u is universal in the following sense. For any flat deformation \mathcal{X}_n of X over an Artin local k -algebra R_n of exponent n , there is a ring homomorphism $\alpha_n : R_n^u \rightarrow R_n$ such that \mathcal{X}_n is the pull-back of X by α_n .

Remark. The condition $H^0(\Theta_X) = 0$ ensures the existence of moduli space of deformations of X , and what we need is in fact the latter condition. Thus one can consider a weaker condition than $H^0(\Theta_X) = 0$ but we omit the details.

The proof will be given in §3.4. We also have the following result on the differential operators. Let us denote by $\mathcal{D}iff_Y$ the sheaf of \mathcal{O}_Y -differential operators for a scheme Y . $\mathcal{D}iff_Y^{\leq n}$ denotes the subsheaf of order $\leq n$.

Theorem 3.1.2. We have a natural morphism

$$\kappa^{\leq n} : \mathcal{D}iff_{S_n^u}^{\leq n} \longrightarrow \mathcal{O}_{S_n^u} \oplus \mathcal{H}^0(J_n)$$

of cocommutative dg coalgebras over $S_n^u := \text{Spec}(R_n^u)$. Therefore, we also have a morphism

$$\kappa : \mathcal{D}iff_{S^u} \longrightarrow \mathcal{O}_{S^u} \oplus \mathcal{H}^0(J).$$

By universality we have

Corollary ([HS97]). For any flat deformation \mathcal{X}_n of X over an Artin local k -algebra R_n of exponent n , we have a natural morphism

$$\mathcal{D}iff_{S_n}^{\leq n} \longrightarrow \mathcal{O}_{S_n} \oplus \mathcal{H}^0(\alpha_n^* J_n)$$

of commutative dg coalgebras over $S_n := \text{Spec}(R_n)$, where $\alpha_n : S_n \rightarrow \text{Spec}(R_n^u)$ is the morphism in Theorem 3.1.1 (2).

The case $n = 1$ coincides with the classical Kodaira-Spencer map

$$\mathcal{D}iff_{S_1}^{\leq 1} / \mathcal{O}_{S_1} = \Theta_{S_1} \longrightarrow \mathcal{H}^0(\alpha_1^* J_1) \simeq R\pi_*^1 \Theta_{\mathcal{X}_1/S_1},$$

where $\pi : \mathcal{X}_1 \rightarrow S_1$ is the canonical projection. Hence we call $\kappa^{\leq n}$ the *higher Kodaira-Spencer map*.

Remark. A similar result as Theorem 3.1.2 is shown in [R06, §3]. The construction of higher Kodaira-Spencer maps using Ran space was originally announced in [R93]. A similar construction without Ran space was given by [EV94].

Before starting the proof of Theorem 3.1.1, we give two preparations in §3.2 and §3.3.

3.2. Maurer-Cartan equation. Let us briefly recall the Maurer-Cartan equation in dg Lie algebras. For a dg k -Lie algebra $\mathfrak{g} = (\mathfrak{g}^\bullet, d, [\cdot, \cdot])$, consider a solution $\alpha \in \mathfrak{g}^1$ of the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

We have

Fact 3.2.1. The twisted differential

$$d_\alpha := d + [\alpha, -]$$

gives a new dg k -Lie algebra

$$\mathfrak{g}_\alpha := (\mathfrak{g}^\bullet, d_\alpha, [\cdot, \cdot]).$$

3.3. Resolution. In [R00, §4] the universal flat deformation of a complex manifold X is constructed over the algebra $R^u = R^u(\Theta_X)$. The construction is done in terms of the Jacobi complex $J(\mathcal{Q}(\Theta_X))$ with $\mathcal{Q}(\Theta_X)$ the Čech or Dolbeault resolution of Θ_X .

Here we spell out a construction of resolution via the Thom-Sullivan complex formalism following [BD04, §4.1.3, Proof of Lemma] and [HS97, §5.2].

For a finite set I , denote by \mathbb{A}_I the subscheme of the affine space \mathbb{A}^I defined by the equation $\sum t_i = 1$. Thus

$$\mathbb{A}_I = \text{Spec}(R_I), \quad R_I := k[t_i \mid i \in I]/(\sum t_i - 1).$$

Denote by Ω_I the algebraic de Rham algebra of R_I over k .

$$\Omega_I := \Gamma(\mathbb{A}_I, \Omega_{\mathbb{A}_I/k}) \simeq R_I[dt_i \mid i \in I]/(\sum dt_i).$$

It is a commutative dg k -algebra with $\deg(dt_i) = 1$ and $d(t_i) = dt_i$, $d(dt_i) = 0$. For each $K \subset I$ we have a natural projection $\psi : \Omega_I \twoheadrightarrow \Omega_K$.

Let X be a separated quasi-compact k -scheme. Take a finite affine covering $\mathcal{U} = \{U_s\}_{s \in S}$ of X . For $I \subset S$ denote by

$$j_I : U_I := \cap_{i \in I} U_i \hookrightarrow X$$

the corresponding embedding. For each $K \subset I$ we have a diagram

$$j_{K*} \mathcal{O}_{U_K} \otimes \Omega_K \xrightarrow{\varphi \otimes \text{id}} j_{I*} \mathcal{O}_{U_I} \otimes \Omega_K \xleftarrow{\text{id} \otimes \psi} j_{I*} \mathcal{O}_{U_I} \otimes \Omega_I.$$

Now let us introduce

Definition. Define the subalgebra $\mathcal{Q}_{\mathcal{U}}$ of the commutative dg \mathcal{O}_X -algebra $\prod_I j_{I*} \mathcal{O}_{U_I} \otimes \Omega_I$ to be

$$\mathcal{Q}_{\mathcal{U}} := \{(f_I) \mid (\varphi \otimes \text{id})(f_K) = (\text{id} \otimes \psi)(f_I) \text{ for any } K \subset I\} \subset \prod_I j_{I*} \mathcal{O}_{U_I} \otimes \Omega_I.$$

$\mathcal{Q}_{\mathcal{U}}$ is a *Dolbeault \mathcal{O}_X -algebra* in the sense of [BD04, §4.1.3]. Namely, it is a commutative unital dg \mathcal{O}_X -algebra, quasi-coherent as an \mathcal{O}_X -module, satisfying

- (1) the structure map $\mathcal{O}_X \rightarrow \mathcal{Q}_{\mathcal{U}}$ as an \mathcal{O}_X -algebra is a quasi-isomorphism,
- (2) $\mathcal{Q}_{\mathcal{U}}$ is homotopically \mathcal{O}_X -flat, namely, for every acyclic complex F of \mathcal{O}_X -modules the complex $\mathcal{Q}_{\mathcal{U}} \otimes_{\mathcal{O}_X} F$ is acyclic,
- (3) $\text{Spec}(\mathcal{Q}_{\mathcal{U}}^0)$ is an affine scheme.

By this remark we have

Lemma. For a quasi-coherent \mathcal{O}_X -module M , the canonical map $M \rightarrow M \otimes \mathcal{Q}_{\mathcal{U}}$ is a quasi-isomorphism. In particular, the class of $M \otimes \mathcal{Q}_{\mathcal{U}}$ in the derived category is independent of the choice of \mathcal{U} .

Remark. (1) Let us mention that in [HS97, §5.2] the same construction is given in terms of the injective limit in the category of the diagrams. In [HS97] $\mathcal{Q}_{\mathcal{U}}$ is used for the construction of higher Kodaira-Spencer maps.
 (2) If X is smooth and proper over $k = \mathbb{C}$, then the classical Dolbeault algebra $\mathcal{Q}_{\overline{\partial}}$, namely the $\overline{\partial}$ -resolution of holomorphic functions

$$\mathcal{Q}_{\overline{\partial}} := (\Omega_X^{0,0} \xrightarrow{\overline{\partial}} \Omega_X^{0,1} \xrightarrow{\overline{\partial}} \dots)$$

is clearly a Dolbeault \mathcal{O}_X -algebra except the quasi-coherent property. In [BD04, §4.1.4] such an object is called a Dolbeault-style algebra. In [R00] $\mathcal{Q}_{\overline{\partial}}$ is used in the construction of the universal deformation.

3.4. The construction of universal family. Let us start the proof of Theorem 3.1.1. following [R00]. Actually the strategy is based on the classical Kodaira-Spencer theory [KS58].

By the assumption on X we can take a finite affine covering $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in A}$ of X . Set

$$\mathfrak{g}_{\mathcal{U}} := \mathcal{Q}_{\mathcal{U}} \otimes \Theta_X = (\mathfrak{g}_{\mathcal{U}}^{\bullet}, d_{\mathfrak{g}}, [\cdot, \cdot]_{\mathfrak{g}}), \quad (3.4.1)$$

where $\mathcal{Q}_{\mathcal{U}}$ is given in §3.3. $\mathfrak{g}_{\mathcal{U}}$ is nothing but the Čech resolution of Θ_X . It is a dg Lie algebra satisfying the conditions in Definition 2.3.3. By the discussion at Lemma 2.3.5, we have the Jacobi complex $J_n(\mathfrak{g}_{\mathcal{U}})$ and

$$J_n(\mathfrak{g}_{\mathcal{U}}) \xrightarrow{\sim} J_n.$$

Assume that we are given a flat deformation \mathcal{X}_n of X over an Artin k -algebra R_n of exponent n . Denote the maximal ideal of R_n by \mathfrak{m}_n . We have $\mathfrak{m}_n^{n+1} = 0$.

Denoting $\mathcal{O} := \mathcal{O}_X$ and $\mathcal{O}_n := \mathcal{O}_{\mathcal{X}_n}$, we have a set $\{\psi_{\alpha}^n\}_{\alpha \in A}$ of isomorphisms of algebras

$$\psi_{\alpha}^n : \mathcal{O}_n(U_{\alpha}^n) \xrightarrow{\sim} \mathcal{O}(U_{\alpha}) \otimes R_n.$$

Here U_{α}^n is the open subset of \mathcal{X}_n corresponding to U_{α} . Then we can find $s_{\alpha} \in \mathfrak{g}_{\mathcal{U}}^0(U_{\alpha}) \otimes \mathfrak{m}_n$ such that

$$\exp(s_{\alpha}) = \psi_{\alpha}^n \circ C,$$

where $C : X \times_{\text{Spec}(k)} \text{Spec}(R_n) \rightarrow \mathcal{X}_n$ is a global trivialization and $\exp(s_{\alpha}) = \sum_{i=0}^n s_{\alpha}^i / i!$ is the formal exponential. Then on $U_{\alpha, \beta} := U_{\alpha} \cap U_{\beta}$ the cocycle

$$D_{\alpha, \beta}^n := \psi_{\alpha}^n (\psi_{\beta}^n)^{-1} \in \text{Aut}_{R_n}(\mathcal{O}(U_{\alpha, \beta}) \otimes R_n)$$

is expressed by

$$D_{\alpha,\beta}^n = \exp(s_\alpha) \exp(-s_\beta).$$

Below we denote by $s := (s_\alpha) \in \mathfrak{g}_U^0 \otimes \mathfrak{m}_n = \Theta_X \otimes \mathfrak{m}_n$.

Recall (2.1.7) that the differential d of the complex $J_n(\mathfrak{g}_U)$ is given by $d = d' + d''$, where d' comes from the differential $d_{\mathfrak{g}}$ of the Čech resolution, and d'' comes from the Lie bracket of Θ_X . Define

$$u_n := \exp(-s)d'(\exp(s)) = -d'(\exp(-s))\exp(s).$$

The cocycle condition $D_{\alpha\beta}^n D_{\beta\gamma}^n = D_{\alpha\gamma}^n$ yields the Maurer-Cartan equation

$$d'(u_n) + \frac{1}{2}[u_n, u_n]_{\mathfrak{g}} = 0.$$

Now set

$$v_n := (u_n, \frac{1}{2}(u_n)^2, \dots, \frac{1}{n!}(u_n)^n) \in J_n(\mathfrak{g}_U) \otimes \mathfrak{m}_n.$$

One can easily check that the cohomology class

$$[v_n] \in \Gamma(X^S, J_n(\mathfrak{g}_U)) \otimes \mathfrak{m}_n \simeq \Gamma(X^S, J_n) \otimes \mathfrak{m}_n$$

depends only on the fixed deformation \mathcal{X}_n , namely is independent of \mathcal{U} , $\{\psi_\alpha^n\}$ and C . Thus we have constructed a correspondence

$$\mathcal{X}_n / \text{Spec}(R_n) \mapsto [v_n] \in \Gamma(X^S, J_n) \otimes \mathfrak{m}_n.$$

Conversely, starting from a cohomology class $[v] \in \Gamma(X^S, J_n) \otimes \mathfrak{m}_n$, we can construct a flat deformation as follows. Take a representative v_n of $[v_n]$ and define $u_n \in J_n^1$ to be the degree 1 of v_n . It satisfies the Maurer-Cartan equation. Then by Fact 3.2.1 we can modify the dg Lie algebra (3.4.1) to

$$\mathfrak{g}'_U = (\mathfrak{g}'_U, d_u, [\cdot, \cdot]_{\mathfrak{g}}), \quad d_u := d_{\mathfrak{g}} + [u, -].$$

Then

$$\mathcal{O}_n := \ker(d_u : \mathfrak{g}'_U \otimes R_n \rightarrow \mathfrak{g}'_U \otimes R_n)$$

is a sheaf of R_n -algebras. It is flat over R_n by the same reason as [R00, Lemma 4.1]. Thus we have a flat deformation

$$\mathcal{X}_n := \text{Spec}(\mathcal{O}_n).$$

The universal family X_n^u is obtained by applying the discussion to $\text{id} : R_n^u \xrightarrow{\sim} R_n^u$. The construction respects the filtration J_\bullet of J , so we have the limit universal family X^u over $\varprojlim \text{Spec } R_n^u = \text{Spf } R^u$.

Remark. In the argument of [R00] an emphasis is put on the OS structure, which consists of the data on a coalgebra C equivalent to the filtration by maximal ideals on the dual Artin algebra C^* . In our formulation, this structure is already built in the tensor structure $\mathcal{CM}_{\mathcal{O}}(X^{\mathfrak{G}})^*$ where our coalgebra J_n lives. See also Remark 1.3.3.

Before turning to the proof of Theorem 3.1.2, we give a preparation in the next §3.5.

3.5. Connecting morphism. Let us give a detailed explanation of *unital* coalgebra. An element u of a k -coalgebra $C = (C, \Delta, \varepsilon)$ is called *group-like* if $du = 0$, $\Delta(u) = u \otimes u$ and $\varepsilon(u) = 1 \in k$. A group-like element u defines a splitting $C = ku \oplus C^+$ with $C^+ := \ker(\varepsilon)$. Denote by $\pi_u : C \rightarrow C^+$ the projection. Such u also defines a filtration $F_\bullet^u C$ by $F_{n-1}^u C := \ker(C \rightarrow C^{\otimes n} \rightarrow (C^+)^{\otimes n})$, where the first map is the n -th composition of Δ , and the second one is $\pi_u^{\otimes n}$. u is called a *unit* if the filtration $F_\bullet^u C$ is exhaustive. A unital coalgebra is a pair $(C, 1_C)$ of coalgebra and its unit. Similarly one can define a unital coalgebra in any unital abelian k -linear tensor category \mathcal{M} .

In particular, taking $\mathcal{M} = \mathcal{CM}_{\mathcal{O}}(X)^*$ with X a k -scheme, we denote by $\mathcal{Cocom}_u(X)$ the category of unital cocommutative dg \mathcal{O}_X -coalgebras. Let \mathfrak{g} be a dg Lie \mathcal{O}_X -algebra and $C(\mathfrak{g})$ be its Chevalley complex. Since the coproduct on $C(\mathfrak{g})$ is cocommutative and $1 \in C(\mathfrak{g})^0 = \mathcal{O}_X$ is a unit, we have $C(\mathfrak{g}) \in \mathcal{Cocom}_u(X)$. For a morphism $f : C \rightarrow C(\mathfrak{g})$ of dg \mathcal{O}_X -coalgebras, we denote by $f_i := p_i \circ f$ the composition with the projection $p_i : C(\mathfrak{g}) \rightarrow \text{Sym}^i(\mathfrak{g}[1])$.

Fact 3.5.1 ([Q69, Appendix B, 5.3]). We have a bijection

$$\text{Hom}_{\mathcal{Cocom}_u(X)}(C, C(\mathfrak{g})) \xrightarrow{\sim} \text{MC}(C, \mathfrak{g}), \quad f \mapsto f_1,$$

where the target is the space of solutions of Maurer-Cartan equation.

$$\text{MC}(C, \mathfrak{g}) := \{f_1 \in \text{Hom}_{\mathcal{CM}_{\mathcal{O}}(X)}(C, \mathfrak{g}[1]) \mid df_1 + \frac{1}{2}[f_1, f_1] = 0, f_1(1_C) = 0\}.$$

Next we recall the connecting morphism of Lie algebras following [HS97, §2.3].

Fix a unital abelian k -linear tensor category \mathcal{M} . Let \mathfrak{g} be a dg Lie algebra in \mathcal{M} and $\mathfrak{h} \subset \mathfrak{g}$ be a dg Lie ideal. Denote by $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ the injection and set $\mathfrak{C} := \text{Cone}(\varphi)$. Thus \mathfrak{C} is a complex with

$$\mathfrak{C}^n = \mathfrak{h}^{n+1} \oplus \mathfrak{g}^n, \quad d_{\mathfrak{C}}(x, y) = (-d_{\mathfrak{h}}x, \varphi(x) + d_{\mathfrak{g}}y),$$

where $d_{\mathfrak{C}}$, $d_{\mathfrak{g}}$ and $d_{\mathfrak{h}}$ are the differentials of \mathfrak{C} , \mathfrak{g} and \mathfrak{h} . \mathfrak{C} is a dg Lie algebra by

$$[(x, y), (x', y')] := ((-1)^p[y, x'] + [x, y'], [y, y']), \quad x, x' \in \mathfrak{h}, y \in \mathfrak{g}, y \in \mathfrak{g}^p.$$

Define the morphisms ψ, π of \mathbb{Z} -graded objects by

$$\psi : \mathfrak{C} \longrightarrow \mathfrak{h}[1], \quad (x, y) \longmapsto x; \quad \pi : \mathfrak{C} \longrightarrow \mathfrak{g}, \quad (x, y) \longmapsto y.$$

Note that ψ is a dg morphism but π is not. Denote by $T(V) = \bigoplus_{n \geq 0} V^{\otimes_{\mathcal{M}} n}$ the tensor algebra of a complex V . Define the morphism $\tilde{c} : T(\mathfrak{C}) \rightarrow \mathfrak{g}[1]$ of \mathbb{Z} -graded objects inductively by

$$\tilde{c}|_{T^0(\mathfrak{C})} := 0, \quad \tilde{c}|_{T^1(\mathfrak{C})} := \psi,$$

and for $u \in T^n(\mathfrak{C})$ and $x \in \mathfrak{C}$ by

$$\tilde{c}(xu) := (-1)^{|x|}[\pi(x), \tilde{c}(u)].$$

Then by [HS97, §2.3.3 Theorem] \tilde{c} factors through the enveloping algebra $U(\mathfrak{C})$ of the Lie algebra \mathfrak{C} , and the obtained morphism $U(\mathfrak{C}) \rightarrow \mathfrak{h}[1]$ satisfies the Maurer-Cartan equation. Thus by Fact 3.5.1 we have a morphism

$$c : U(\mathfrak{C}) \longrightarrow C(\mathfrak{h})$$

of dg coalgebras to the Chevalley complex of \mathfrak{h} . Since $U(\mathfrak{C}) \simeq U(\mathfrak{g}/\mathfrak{h})$ as a dg coalgebra, we have in total

Fact 3.5.2 ([HS97, §2.3, §3.3]). For a dg Lie algebra \mathfrak{g} and dg Lie ideal $\mathfrak{h} \subset \mathfrak{g}$, there is a morphism

$$c : U(\mathfrak{g}/\mathfrak{h}) \longrightarrow C(\mathfrak{h})$$

of unital cocommutative dg coalgebras in \mathcal{M} . It is called the *connecting morphism* of the pair $\mathfrak{h} \subset \mathfrak{g}$.

By the construction, the first order part c^1 of c is the coboundary map in the long exact sequence

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \xrightarrow{c^1} \mathfrak{h}[1] \longrightarrow \cdots,$$

which is the origin of the name ‘connecting morphism’.

Remark. Instead of the assumption that \mathfrak{h} is a dg Lie ideal, one may ask whether there is a dg Lie algebra structure on $\text{Cone}(\varphi)$ of general dg Lie algebra morphism φ . [FM07] gives the answer that $\text{Cone}(\varphi)$ has no dg Lie algebra structure but has a natural L_∞ -structure. Note also that [R06, R08] discussed a similar situation called Lie atoms.

3.6. The construction of higher Kodaira-Spencer maps. Let us give a proof of Theorem 3.1.2. We start with the remark that the case $n = 1$ recovers the classical Kodaira-Spencer theory by the universality property. Namely, given a first order deformation \mathcal{X}_1 of X over $S_1 = \text{Spec}(R_1)$, we have the Kodaira-Spencer map

$$\Theta_{S_1} \longrightarrow R\pi_*^1 \Theta_{\mathcal{X}_1/S_1} \tag{3.6.1}$$

with $\pi : \mathcal{X}_1 \rightarrow S_1$ the projection. By the classical theory, we know that this map coincides with the coboundary map c^1 in

$$0 \longrightarrow \pi_* \Theta_{\mathcal{X}_1} \longrightarrow \pi_* \Theta_{\mathcal{X}_1} \longrightarrow \pi_* \pi^* \Theta_{S_1} \simeq \Theta_{S_1} \xrightarrow{c^1} R\pi_*^1 \Theta_{\mathcal{X}_1/S_1} \longrightarrow \cdots$$

induced from the short exact sequence

$$0 \longrightarrow \Theta_{\mathcal{X}_1/S_1} \longrightarrow \Theta_{\mathcal{X}_1} \longrightarrow \pi^* \Theta_{S_1} \longrightarrow 0.$$

Note also $(R\pi_*^1 \Theta_{\mathcal{X}_1/S_1})_p \simeq H^1(X, \Theta_X) \simeq \Gamma(X^\oplus, J_n^1)$, where $p \in S_1$ corresponds to the original X . (J_n^1 is the degree 1 part of the complex J_n .) Thus we have

$$R\pi_*^1 \Theta_{\mathcal{X}_1/S_1} \xrightarrow{\sim} \mathcal{H}^0(J_n^1 \otimes R_1) = \mathcal{H}^0(\alpha_1^* J_n^1) \tag{3.6.2}$$

for any $n \in \mathbb{Z}_{\geq 1}$. Putting (3.6.1) and (3.6.2) together, we obtain

$$\mathcal{D}iff_{S_1}^{\leq 1} = \mathcal{O}_{S_1} \oplus \Theta_{S_1} \longrightarrow \mathcal{O}_{S_1} \oplus \mathcal{H}^0(\alpha_1^* J_n^1).$$

Now let us construct a higher analog. Let $\pi : \mathcal{X}_n^u \rightarrow S_n^u = \text{Spec}(R_n^u)$ be the universal deformation of order n . Applying Fact 3.5.2 to $\pi_* \Theta_{\mathcal{X}_n^u/S_n^u} \longrightarrow \pi_* \Theta_{\mathcal{X}_n^u}$ and $\mathcal{M} = \mathcal{M}_{\mathcal{O}}(S_n^u)$, we have a morphism

$$c^{\leq n} : U(\Theta_{S_n^u}) \longrightarrow C(\pi_* \Theta_{\mathcal{X}_n^u/S_n^u}).$$

On the other hand, we have

$$U(\Theta_{S_n^u}) \xrightarrow{\sim} \mathcal{D}iff^{\leq n}(S_n^u), \quad C(\pi_* \Theta_{\mathcal{X}_n^u/S_n^u}) \xrightarrow{\sim} \mathcal{O}_{S_n^u} \oplus \mathcal{H}^0(J_n).$$

Thus $c^{\leq n}$ gives the desired morphism

$$\kappa^{\leq n} : \mathcal{D}iff^{\leq n}(S_n^u) \longrightarrow \mathcal{O}_{S_n^u} \oplus \mathcal{H}^0(J_n).$$

4. THE MODULI SPACE OF G -BUNDLES

We address an analog of the higher Kodaira-Spencer map in §3 for the moduli space of G -bundles with G an algebraic group. Our strategy basically follows [HS97].

4.1. Lie algebroid. Let X be a smooth scheme over k . A (dg) *Lie algebroid over X* (or (dg) *Lie \mathcal{O}_X -algebroid*) is a sheaf \mathcal{L} of (dg) Lie k -algebras on X together with a structure of a left \mathcal{O}_X -module and a morphism $\tau : \mathcal{L} \rightarrow \Theta_X = \text{Der}_k(\mathcal{O}_X)$ of (dg) Lie k -algebras and \mathcal{O}_X -modules such that $[a, fb] = f[a, b] + \tau(a)(f)b$ for any $a, b \in \mathcal{L}$ and $f \in \mathcal{O}_X$. The morphism τ is called the *anchor* of \mathcal{L} .

For a Lie \mathcal{O}_X -algebroid \mathcal{L} , a left \mathcal{L} -module is an \mathcal{O}_X -module M with an action of \mathcal{L} as a Lie k -algebra with compatibility condition $l(fm) = l(f)m + f(lm)$ and $(fl)m = f(lm)$ for any $f \in \mathcal{O}_X$, $m \in M$ and $l \in \mathcal{L}$. The dg version is similarly defined.

For a Lie algebroid over X , denote by $U_{\mathcal{O}_X}(\mathcal{L})$ the *twisted enveloping algebra*. Let us recall its definition. Denote by $U_k(\mathcal{L})^+$ the augmented ideal of the universal enveloping algebra $U_k(\mathcal{L})$ of \mathcal{L} as a k -Lie algebra. Define $U_{\mathcal{O}_X}(\mathcal{L})^+$ to be the quotient of $U_k(\mathcal{L})^+$ by the two-sided ideal generated by $a \cdot fb - fa \cdot b - \tau(a)(f)b$ for all $a, b \in \mathcal{L}$ and $f \in \mathcal{O}_X$. Then $U_{\mathcal{O}_X}(\mathcal{L}) := \mathcal{O}_X \oplus U_{\mathcal{O}_X}(\mathcal{L})^+$ with the unital algebra structure given by $f \cdot a = fa$ and $a \cdot f = fa + \tau(a)(f)$ for $a \in \mathcal{L}$ and $f \in \mathcal{O}_X$.

$U_{\mathcal{O}_X}(\mathcal{L})$ has a filtration $F_\bullet U_{\mathcal{O}_X}(\mathcal{L})$ coming from the standard one on $U_k(\mathcal{L})$. It also has a coalgebra structure induced by that on $U_k(\mathcal{L})$.

Example 4.1.1. We can take $\mathcal{L} = \Theta_X$. Then a left \mathcal{L} -module is nothing but a left \mathcal{D}_X -module. We also have $U_{\mathcal{O}_X}(\Theta_X) = \mathcal{D}iff_X$, the sheaf of \mathcal{O}_X -differential operators, and $F_n U_{\mathcal{O}_X}(\Theta_X) = \mathcal{D}iff_X^{\leq n}$.

4.2. Higher Kodaira-Spencer map for Lie algebroid. Let us explain the construction of higher Kodaira-Spencer maps by [HS97, §7.1] with the help of Jacobi complexes.

Let $\pi : \mathcal{X} \rightarrow S$ be a smooth separated map of schemes over k . We have the short exact sequence

$$0 \longrightarrow \Theta_{\mathcal{X}/S} \longrightarrow \Theta_{\mathcal{X}} \xrightarrow{\varepsilon} \pi^* \Theta_S \longrightarrow 0$$

Denote by π^{-1} the functor of set-theoretical inverse image. Hence $\pi^{-1} \Theta_S \subset \pi^* \Theta_S$ is a Lie $\pi^{-1} \Theta_S$ -algebra. Set

$$\Theta_\pi := \varepsilon^{-1}(\pi^{-1} \Theta_S) \subset \Theta_{\mathcal{X}}, \quad (4.2.1)$$

which is the sheaf of vector fields on \mathcal{X} preserving π . we have a short exact sequence

$$0 \longrightarrow \Theta_{\mathcal{X}/S} \longrightarrow \Theta_\pi \xrightarrow{\varepsilon} \pi^{-1} \Theta_S \longrightarrow 0 \quad (4.2.2)$$

of Lie k -algebras and $\pi^{-1} \Theta_S$ -modules.

Let \mathcal{A} be a dg Lie algebroid over \mathcal{X} such that the anchor $\tau : \mathcal{A} \rightarrow \Theta_{\mathcal{X}}$ is an epimorphism, namely, the zeroth part $\tau^0 : \mathcal{A}^0 \rightarrow \Theta_{\mathcal{X}}$ is surjective. Setting $\mathcal{A}_{\mathcal{X}/S} := \tau^{-1}(\Theta_{\mathcal{X}/S})$ and $\mathcal{A}_\pi := \tau^{-1}(\Theta_\pi)$, we have a short exact sequence

$$0 \longrightarrow \mathcal{A}_{\mathcal{X}/S} \longrightarrow \mathcal{A}_\pi \longrightarrow \pi^{-1} \Theta_S \longrightarrow 0 \quad (4.2.3)$$

of Lie k -algebras and $\pi^{-1} \Theta_S$ -modules.

Applying the functor $J(-)$ to the exact sequence (4.2.3), we have

$$0 \longrightarrow J(\mathcal{A}_{\mathcal{X}/S}) \longrightarrow J(\mathcal{A}_\pi) \longrightarrow J(\pi^{-1} \Theta_S) \longrightarrow 0 \quad (4.2.4)$$

of complexes over \mathcal{X}^S . On the other hand, we have a canonical adjunction map

$$\Theta_S \longrightarrow J(\pi^{-1} \Theta_S).$$

Taking the pull-back of (4.2.4) by this adjunction map, we have a short exact sequence

$$0 \longrightarrow J(\mathcal{A}_{\mathcal{X}/S}) \longrightarrow \overline{\mathcal{A}}_\pi \longrightarrow \Theta_S \longrightarrow 0$$

with

$$\overline{\mathcal{A}}_\pi := J(\mathcal{A}_\pi) \otimes_{J(\pi^{-1} \Theta_S)} \Theta_S.$$

$J(\mathcal{A}_\pi)$ and Θ_S induce on $\overline{\mathcal{A}}_\pi$ the structure of dg Lie k -algebra and \mathcal{O}_X -module. Thus we have the pair

$$J(\mathcal{A}_{\mathcal{X}/S}) \hookrightarrow \overline{\mathcal{A}}_\pi$$

of a dg Lie \mathcal{O}_X -algebra and its dg Lie ideal.

Then we can apply the construction in §3.5 of the connecting morphism to this pair. The discussion in §3.4 can also be applied to the present situation if the universal family exists, and we have a description of the universal family of deformations of $\mathcal{A}_{\mathcal{X}/S}$.

4.3. The G -bundle case. Let X be a smooth k -scheme, G be a semi-simple algebraic k -group, and $p : P \rightarrow X$ be a G -torsor over X . Set

$$\mathcal{A}_P := (p^{-1} \Theta_X)^G,$$

namely \mathcal{A}_P is the sheaf such that $\mathcal{A}_P(U)$ is the space of G -invariant vector fields on $p^{-1}(U)$. p induces a surjection $\tau : \mathcal{A}_P \rightarrow \Theta_X$, which makes \mathcal{A}_P a Lie algebroid over X . If $H^0(X, \mathcal{A}_P) = 0$ and there is a universal deformation of (X, P) .

Now the argument in the previous subsection can be applied to the Lie algebroid \mathcal{A}_P .

Theorem 4.3.1 ([HS97]). Assume $H^0(X, \mathcal{A}_P) = 0$. Set

$$R_n^u := k \oplus \Gamma(\mathcal{R}(X), J_n(\mathcal{A}_P)).$$

- (1) For each $n \in \mathbb{Z}_{\geq 1}$, there is a flat deformation \mathcal{Y}_n^u of (X, P) over $\mathrm{Spec}(R_n^u)$. The data $\{(\mathcal{Y}_n^u, R_n^u)\}_{n \geq 1}$ form a direct system with the limit (\mathcal{Y}^u, R^u) , a flat formal deformation \mathcal{P}^u over $\mathrm{Spf}(R^u)$. Moreover \mathcal{Y}_n^u is universal in the sense of Theorem 3.1.1 (2).
- (2) We have a natural morphism

$$\kappa^{\leq n} : \mathcal{D}iff_{\tilde{S}_n^u}^{\leq n} \longrightarrow \mathcal{O}_{S_n^u} \oplus \mathcal{H}^0(J_n(\mathcal{A}_P))$$

of cocommutative dg coalgebras over $S_n^u := \mathrm{Spec}(R_n^u)$.

We have an X -fixed version, namely the deformation problem of the G -torsors over a fixed scheme X . Then we can take the Jacobi complex $J(\mathfrak{g}_P)$ of

$$\mathfrak{g}_P := \ker(\tau : \mathcal{A}_P \longrightarrow \Theta_X),$$

and apply the same argument.

5. FLAT CONNECTION ON THE JACOBI HOMOLOGY

In [R06] Ran gave a general construction of (projective) flat connections related to the universal deformation. His strategy can be stated in the following steps.

- (1) Translate the heat equation in Hitchin's construction [H90] of projective flat connections into the language of dg Lie algebras on the Ran space. As a result one obtains the “connection algebra”, which is the Jacobi complex of a certain Lie atom.
- (2) The connection algebra has a canonical trivialization over the universal deformation algebra. This trivialization gives the desired connections.

In this note we give another approach. Our strategy is to take an analog of flat connections on chiral homology due to Beilinson and Drinfeld in their theory of chiral algebra [BD04, §4]. Let us briefly explain their argument.

Chiral algebras are certain Lie algebra objects in a non-standard tensor category of \mathcal{D} -modules on a fixed curve X . One can consider a kind of reduced Chevalley complex $\overline{\mathcal{C}}(A)$ (called the Chevalley-Cousin complex) of a chiral algebra A . $\mathcal{C}(A)$ can be seen as a \mathcal{D} -module on the Ran space $\mathcal{R}(X)$. Chiral homology $C^{\mathrm{ch}}(A)$ is defined to be $R\Gamma(\mathcal{R}(X), \mathrm{DR}(\mathcal{C}(A)))$, where $\mathrm{DR}(-)$ is the de Rham functor on $\mathcal{R}(X)$ (we omit the precise definition). $C^{\mathrm{ch}}(A)$ is a generalization of the conformal block in conformal field theory.

The main ingredient of their argument is the BV algebra structure on the Chevalley complex $\overline{\mathcal{C}}(A)$, which yields a canonical trivialization on Lie algebra action on $\overline{\mathcal{C}}(A)$. A certain set of assumptions called “the package” in [BD04, §4.5] enables one to construct flat connections on $C^{\mathrm{ch}}(A)$ using this canonical trivialization.

One finds that their situation looks similar to ours, namely they handle Lie algebra objects on the Ran space. Moreover, the BV algebra structure exists for the Chevalley complex of Lie algebra in any tensor category, as we explain in §5.1. Thus it is natural to expect that one can take an analog of the arguments in [BD04].

5.1. BV algebras. Following [BD04, 4.1.6], we recall the notion of *Batalin-Vilkovisky algebras*, or *BV algebras* for short. See also [LV12, §13.7], although its presentation has some minor difference from ours.

Roughly speaking, a BV k -algebra C is a 1-Poisson (or Gerstenhaber) dg k -algebra. Since we will later consider BV algebras in several different tensor categories, let us spell out the definition using the language of operads.

Definition. The *BV operad* \mathcal{BV} is defined to be a dg k -operad which is inhomogeneous quadratic (in the sense of [LV12, §7.8]) generated by the differential d of degree 1, the product $m = (- \cdot -)$ of degree 0 and the 1-Poisson bracket $c = \{-, -\}$ of degree 1. The relations consists of $d^2 = 0$, the 1-Poisson operad relation for m and c , and the $c = d \circ m$.

Let \mathcal{M} be an abelian k -linear tensor category with the tensor structure $\otimes_{\mathcal{M}}$. We denote by \mathcal{CM} the dg category of complexes in \mathcal{M} . The induced tensor product on \mathcal{CM} is denoted by the same symbol $\otimes_{\mathcal{M}}$. Using the framework of algebras over operad (see [LV12, Chap. 5] for example), we have

Definition. A *BV algebra* in \mathcal{M} means a complex C in \mathcal{M} together with a morphism $\mathcal{BV} \rightarrow \mathcal{E}nd_C$ of dg operads, where $\mathcal{E}nd_C := \bigoplus_n \mathrm{Hom}_{\mathcal{CM}}(C^{\otimes n}, C)$ is the endomorphism operad on C .

For example, letting \mathcal{M} be the category of k -vector spaces, a BV k -algebra C is a complex (C, d_C) of k -vector spaces together with a Poisson structure on the graded vector space $C[-1]$ consisting of the product \cdot_C and the Poisson bracket $\{, \}_C$. These data should satisfy the relation $\{, \}_C = d_C \cdot_C - \cdot_C d_C \otimes_C$. This is what we mentioned roughly in the beginning.

Let \mathcal{M} be an abelian k -linear tensor category again, and $C = (C, d, m, c)$ be a BV algebra in \mathcal{M} . Then $L := C[-1]$ is naturally a Lie algebra in the category \mathcal{CM} with the Lie bracket c , and the BV structure yields an L -action on C . Then we set

$$L_{\dagger} := \mathrm{Cone}(\mathrm{id}_L) \in \mathcal{CM}, \tag{5.1.1}$$

which is a contractible complex. As a \mathbb{Z} -graded object we have $L_{\dagger} \simeq L[1] \oplus L = C \oplus L$. L_{\dagger} is naturally a Lie algebra in \mathcal{CM} (see §3.5), and the L -action on C extends to the L_{\dagger} -action with the component $L[1] = C$ acting on C by m .

5.2. Filtered BV algebras. Let \mathcal{M} be an abelian k -linear tensor category as before. One can consider \mathcal{BV} as a dg filtered operad with the increasing stupid filtration $\mathcal{BV}_n := \mathcal{BV}^{\geq -n}$. Then one has

Definition. A *filtered BV algebra* in \mathcal{M} is a BV algebra C in \mathcal{M} together with an increasing filtration C_\bullet which is compatible with the BV algebra structure. Denote by $\mathcal{BV}(\mathcal{M})$ the category of filtered BV algebras C in \mathcal{M} such that $C_{-1} = 0$ and $\cup_{n \in \mathbb{Z}} C_n = C$. Also let $\overline{\mathcal{BV}}(\mathcal{M})$ be the full subcategory in $\mathcal{BV}(\mathcal{M})$ consisting of objects C such that $C_0 = 0$.

One can naturally augment \mathcal{BV} and obtains a dg operad \mathcal{BV}_u which encodes the structure of unital BV algebras. In other words, we introduce

Definition. Assume that \mathcal{M} has a unit and is a symmetric monoidal category. We define a *unital BV algebra* C in \mathcal{M} to be a BV algebra (C, d, m, c) in \mathcal{M} having a unit $1 \in C^0$ with respect to m such that $d(1) = 0$. In the filtered setting we assume $1 \in C_0$. The subcategory in $\mathcal{BV}(\mathcal{M})$ consisting of unital filtered BV algebras is denoted by $\mathcal{BV}_u(\mathcal{M})$.

By [BD04, §4.1.7. Proposition], $\mathcal{BV}(\mathcal{M})$, $\overline{\mathcal{BV}}(\mathcal{M})$ and $\mathcal{BV}_u(\mathcal{M})$ are closed model categories with weak equivalences being filtered quasi-isomorphisms and fibrations being morphisms f such that $\text{gr}(f)$ is surjective. Thus we have the corresponding homotopy categories $\mathcal{Ho}\mathcal{BV}(\mathcal{M})$, $\mathcal{Ho}\overline{\mathcal{BV}}(\mathcal{M})$ and $\mathcal{Ho}\mathcal{BV}_u(\mathcal{M})$ (recall Definition 2.2.1).

5.3. Chevalley complex. Let \mathcal{M} be as before. For a filtered BV algebra C in \mathcal{M} , the shifted filter $C_1[-1]$ is naturally a dg Lie algebra in \mathcal{M} with respect to $c = \{, \}$. The correspondence $C \mapsto C_1[-1]$ yields functors

$$\overline{\mathcal{BV}}(\mathcal{M}) \longrightarrow \mathcal{Lie}(\mathcal{CM}), \quad \mathcal{BV}_u(\mathcal{M}) \longrightarrow \mathcal{Lie}(\mathcal{CM}),$$

where $\mathcal{Lie}(\mathcal{CM})$ denotes the category of dg Lie algebras in \mathcal{M} . In the case of $\mathcal{BV}_u(\mathcal{M})$, one should suppose \mathcal{M} to have a unit.

Remark. If we have left adjoints of these functors, then they will give us examples of BV algebras automatically. Indeed by [BD04, §4.1.8] we have left adjoints, and they are nothing but constructing the Chevalley complexes as the title of this subsection implies. Let us give an explanation of the way to find this answer.

Recall that the forgetting functor V_0 from Poisson algebras (say over k) to Lie algebras has a left adjoint S assigning to a Lie algebra L the symmetric algebra $\text{Sym}(L)$ together with the Kostant-Kirillov Poisson bracket. The adjoint pair (S, V_0) is the “classical part” of the following pair (U, V) . V is the forgetting functor from associative algebras to Lie algebras with the same vector space and the commutator as the Lie bracket. U is the functor assigning to a Lie algebra L the universal enveloping algebra $U(L)$. Since associative algebra degenerate to Poisson algebras, and since $U(L)$ is a deformation quantization of $S(L)$, we can say (S, V_0) is the classical part of (U, V) .

Thus one can guess that the desired left adjoints are given by $L \mapsto \text{Sym}(L[1])$, where the shift $[1]$ is necessary because we are considering a 1-Poisson structure. $\text{Sym}(L[1])$ is nothing but the Chevalley complex of L .

By [BD04, §4.1.8], the functors $C \mapsto C_1[-1]$ have left adjoints

$$\overline{C}: \mathcal{Lie}(\mathcal{CM}) \longrightarrow \overline{\mathcal{BV}}(\mathcal{M}), \quad C: \mathcal{Lie}(\mathcal{CM}) \longrightarrow \mathcal{BV}_u(\mathcal{M}),$$

where for $L \in \mathcal{Lie}(\mathcal{CM})$ the corresponding $C(L)$ is given by the Chevalley complex of L , and $\overline{C}(L)$ is the reduced Chevalley complex of L . Thus as \mathbb{Z} -graded objects in \mathcal{M} we have

$$C(L) = \text{Sym}(L[1]) = \oplus_{n \geq 0} \text{Sym}^n(L[1]), \quad \overline{C}(L) = \text{Sym}^{\geq 1}(L[1]).$$

The filtration is given by $C(L)_n := \text{Sym}^n(L[1])$. The differential and the 1-Poisson bracket are determined by the condition that the embedding $L = \text{Sym}^1(L[1])[-1] \hookrightarrow C[-1]$ is a morphism of dg Lie algebras. This BV structure respects the filtration.

These functors preserve filtered quasi-isomorphisms, so that they descent to homotopy categories and yield adjoint pairs

$$\mathcal{Ho}\mathcal{Lie}(\mathcal{M}) \xrightleftharpoons{\overline{C}} \mathcal{Ho}\overline{\mathcal{BV}}(\mathcal{M}), \quad \mathcal{Ho}\mathcal{Lie}(\mathcal{M}) \xrightleftharpoons{C} \mathcal{Ho}\mathcal{BV}_u(\mathcal{M})$$

on homotopy categories.

As a corollary, we find that the Jacobi complex has a BV structure.

5.4. Rigidity. Let X be a separated quasi-projective k -scheme and $\mathfrak{g}, L \in \mathcal{Lie}(X)$ as in Definition 2.3.3.

Suppose L acts on \mathfrak{g} . Then $\Gamma(X, L)$ acts on \mathfrak{g} by derivation, and further $\Gamma(X, L)$ acts on the Jacobi complex $J(\mathfrak{g})$. One can replace \mathfrak{g} and L by their resolutions. For example, take the Čech complex \mathcal{Q} in §3.3 and consider $\mathfrak{g}_{\mathcal{Q}} := \mathfrak{g} \otimes_{\mathcal{O}_X} \mathcal{Q}$, and $L_{\mathcal{Q}} := L \otimes_{\mathcal{O}_X} \mathcal{Q}$ instead of \mathfrak{g} and L . We still have a $\Gamma(X, L_{\mathcal{Q}})$ -action on $J(\mathfrak{g}_{\mathcal{Q}}) = J(\mathfrak{g})_{\mathcal{Q}}$. Since $L_{\mathcal{Q}}$ is canonically identified with L in $\mathcal{Ho}\mathcal{Lie}(X)$, $\Gamma(X, L_{\mathcal{Q}})$ is identified with $R\Gamma(X, L)$. Thus the homotopy Lie algebra $R\Gamma(X, L)$ acts on $J(\mathfrak{g})$.

Now we have an analog of the rigidity property of Chevalley-Cousin complex in [BD04, §4.5.2].

Lemma 5.4.1. Suppose that the L -action comes from a Lie algebra morphism $\iota: L \rightarrow \mathfrak{g}$. Then the $R\Gamma(X, L)$ -action on $J(\mathfrak{g})$ is canonically homotopically trivialized.

Proof. Recall the BV structure on $J(\mathfrak{g})$. The morphism ι yields a morphism

$$\Gamma(X, L_Q) \xrightarrow{\iota} \Gamma(X, \mathfrak{g}_Q) \hookrightarrow (J(\mathfrak{g})_Q)[-1]$$

of Lie algebras. So the Lie algebra $\Gamma(X, L_Q)_\dagger$ (see (5.1.1) and around) acts on $J(\mathfrak{g})$ via the canonical action of $(J(\mathfrak{g})_Q[-1])_\dagger$ given by the BV structure. Since $\Gamma(X, L_Q)_\dagger$ is contractible, we obtain a homotopy from the given action to a trivial action. \square

Let us give a variant of this statement. Suppose that $\mathfrak{g} \in \mathcal{L}ie(X)$ is a central extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 0$$

of dg Lie \mathcal{O}_X -algebras and that $L \in \mathcal{L}ie(X)$ acts on \mathfrak{g} by the Lie homomorphism $\bar{\iota} : L \rightarrow \mathfrak{h}$ and the adjoint action of \mathfrak{h} on \mathfrak{g} . Then we have $R\Gamma(X, L)$ -action on $J(\mathfrak{g})$ similarly as above.

Lemma 5.4.2. In this situation, the $R\Gamma(X, L)$ -action is homotopically equivalent to the multiplication of a character.

Proof. Denote by L^\flat the \mathcal{O}_X -extension of L defined as the pull-back of $\mathfrak{g} \rightarrow \mathfrak{h}$ by ι . The resulting Lie algebra morphism $\varepsilon : L^\flat \rightarrow L$ yields a dg Lie algebra $L^\flat_\dagger := \text{Cone}(\varepsilon)$. As in the previous Lemma 5.4.1, the Lie algebra $\tilde{L} := R\Gamma(X, L^\flat_\dagger)$ acts on $J(\mathfrak{g})$, and it is homotopic to $R\Gamma(X, L)$. In the homotopy category \tilde{L} is equivalent to k . Thus we are done. \square

5.5. Relative rigidity. We want to discuss a relative version of the rigidity property explained in §5.4. Our presentation is an analog of “the package” of Lie algebroid action on a chiral algebra governed by a Lie^{*} algebra action in [BD04, §§4.5.4, 4.5.5].

Let $\pi : \mathcal{X} \rightarrow S$ be a smooth proper flat family of k -schemes. For a point $s \in S$ we denote by \mathcal{X}_s the corresponding fiber. All the notions explained in the previous sections have a natural relative version. For example, the Ran space $\mathcal{R}(\mathcal{X}/S)$ is a space fibered over S with the fiber the usual Ran space $\mathcal{R}(\mathcal{X}_s)$. The symbol $\mathcal{L}ie(\mathcal{X}/S)$ denotes the relative version of Definition 2.3.3. Namely the category of \mathcal{O}_S -flat \mathcal{O}_S -modules \mathfrak{g} such that \mathfrak{g}_s is a dg Lie $\mathcal{O}_{\mathcal{X}_s}$ -algebras for every $s \in S$ satisfying the two conditions similar as in Definition 2.3.3. The objects in $\mathcal{L}ie(\mathcal{X}/S)$ will be called the *flat family of dg Lie algebras on \mathcal{X}/S* . We also have the notion of *flat family of dg Lie algebroids on \mathcal{X}/S* . The *relative Jacobi complex $J(\mathfrak{g})$* for $\mathfrak{g} \in \mathcal{L}ie(\mathcal{X}/S)$ is naturally defined. Finally let us denote by $R\pi_*(\mathcal{R}(\mathcal{X}/S), -)$ the relative version of the functor $R\Gamma(\mathcal{X}, -)$ defined in §1.2.

Let \mathcal{L} be a Lie algebroid over S with $\tau : \mathcal{L} \rightarrow \Theta_S$ the anchor. Define

$$\pi^\sharp \mathcal{L} := \pi^{-1} \mathcal{L} \otimes_{\pi^{-1} \Theta_S} \Theta_\pi.$$

Here Θ_π is the subsheaf in $\Theta_{\mathcal{X}}$ consisting of vector fields preserving $\pi^{-1} \mathcal{O}_S \subset \mathcal{O}_{\mathcal{X}}$ (see also (4.2.1) and (4.2.2)). Hence $\pi^\sharp \mathcal{L}$ is a Lie $\pi^{-1} \mathcal{O}_S$ -algebroid acting on $\mathcal{O}_{\mathcal{X}}$, and sits in the exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Theta_{\mathcal{X}/S} & \longrightarrow & \pi^\sharp \mathcal{L} & \longrightarrow & \pi^{-1} \mathcal{L} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \pi^{-1} \tau \\ 0 & \longrightarrow & \Theta_{\mathcal{X}/S} & \longrightarrow & \Theta_\pi & \longrightarrow & \pi^{-1} \Theta_S \longrightarrow 0 \end{array}$$

Let us also define $\pi^\dagger \mathcal{L}$ to be the push-out of $\mathcal{O}_{\mathcal{X}} \otimes_{\pi^{-1} \mathcal{O}_S} \pi^\sharp \mathcal{L}$ by the product map $\mathcal{O}_{\mathcal{X}} \otimes_{\pi^{-1} \mathcal{O}_S} \Theta_{\mathcal{X}/S} \rightarrow \Theta_{\mathcal{X}/S}$. We have a short exact sequence

$$0 \longrightarrow \Theta_{\mathcal{X}/S} \longrightarrow \pi^\dagger \mathcal{L} \longrightarrow \pi^* \mathcal{L} \longrightarrow 0$$

of $\mathcal{O}_{\mathcal{X}}$ -modules, If M is a left \mathcal{L} -module, then $\pi^* M$ is naturally a left $\pi^\dagger \mathcal{L}$ -module.

Mimicking the setting in [BD04, §4.5.4], let us suppose that the following data is given.

- (a) a dg Lie algebroid \mathcal{L} on S ,
- (b) a Lie $\pi^{-1} \mathcal{O}_S$ -algebra L ,
- (c) a dg Lie $\pi^{-1} \mathcal{O}_S$ -algebroid extension \mathcal{K} of $\pi^\sharp \mathcal{L}$ by L ,
- (d) a section $s : \Theta_{\mathcal{X}/S} \rightarrow \mathcal{K}$,
- (e) a left \mathcal{K} -module structure on L (where \mathcal{K} is seen as a Lie $\pi^{-1} \mathcal{O}_S$ -algebroid),
- (f) a Lie algebroid \mathcal{A} on \mathcal{X}/S which is \mathcal{O}_S -flat.
- (g) a left \mathcal{K} -module structure on \mathcal{A} .
- (h) a morphism $\iota : L \rightarrow \mathcal{A}$ of dg Lie algebras on \mathcal{X}/S .

These should satisfy

- (1) The \mathcal{K} -actions on L and \mathcal{A} are compatible with the Lie brackets on L and \mathcal{A} .
- (2) $\iota : L \rightarrow \mathcal{A}$ commutes with the \mathcal{K} -actions.
- (3) $s(\Theta_{\mathcal{X}/S}) \subset \mathcal{K}$ is a Lie ideal. Hence \mathcal{K} is an extension of $\pi^{-1} \mathcal{L}$ by $\mathcal{K}_0 := L \otimes_k \Theta_{\mathcal{X}/S}$.
- (4) $\Theta_{\mathcal{X}/S} \subset \mathcal{K}_0$ acts on \mathcal{A} and L trivially.
- (5) $L \subset \mathcal{K}_0$ acts on L via the adjoint action, and on \mathcal{A} via ι and the adjoint action.
- (6) The adjoint action of \mathcal{K} on $L \subset \mathcal{K}_0$ coincides with the \mathcal{K} -action on L coming from the left \mathcal{K} -module structure.

Theorem 5.5.1. There is a homotopy left \mathcal{L} -module structure on $R^i\pi_*(\mathcal{R}(\mathcal{X}/S), J(\mathcal{A}))$ for each i . In particular, if \mathcal{L} is a plain (non dg) Lie \mathcal{O}_S -algebroid, then $R^i\pi_*(\mathcal{R}(\mathcal{X}/S), J(\mathcal{A}))$ are left \mathcal{L} -modules.

Proof. By the condition (3) we have a short exact sequence

$$0 \longrightarrow \mathcal{K}_0 = L \otimes \Theta_{\mathcal{X}/S} \longrightarrow \mathcal{K} \longrightarrow \pi^{-1}\mathcal{L} \longrightarrow 0 \quad (5.5.1)$$

The strategy is to obtain a homotopy \mathcal{L} -action from the $R\pi_*\mathcal{K}$ -action (given by the sixth data) by trivializing homotopically the action of $R\pi_*\mathcal{K}_0$ as in Lemma 5.4.1.

Let \mathcal{Q} be a Dolbeault \mathcal{O}_S -algebra equipped with a left $\pi^*\mathcal{L}$ -action (which always exists by [BD04, §4.5.5 Proof (i) (b)]). We can replace $L, \mathcal{L}, \mathcal{A}$ and so on by homotopy Lie algebras (algebroids) $L_{\mathcal{Q}} := L \otimes \mathcal{Q}, \mathcal{L}_{\mathcal{Q}}, \mathcal{A}_{\mathcal{Q}}$ and so on preserving the conditions.

By the condition (4), the \mathcal{K}_0 -action on \mathcal{A} factor through an action of $\tilde{\mathcal{K}}_{\mathcal{Q}} := (\mathcal{K}_0 \otimes \mathcal{Q})/(\Theta_{\mathcal{X}/S} \otimes \mathcal{Q})$. Then the sequence (5.5.1) yields

$$0 \longrightarrow \pi_*L_{\mathcal{Q}} \longrightarrow \pi_*\tilde{\mathcal{K}}_{\mathcal{Q}} \longrightarrow \mathcal{L}_{\mathcal{Q}} \longrightarrow 0$$

The $\tilde{\mathcal{K}}$ -action induces a $\pi_*\tilde{\mathcal{K}}$ -action on $R\pi_*(\mathcal{R}(\mathcal{X}/S), J(\mathcal{A}))$.

Set $L_{\dagger} := \text{Cone}(\text{id} : L \xrightarrow{\sim} L)$, which is a contractible Lie algebra. There is also a natural embedding $L_{\mathcal{Q}} \hookrightarrow L_{\dagger} \otimes \mathcal{Q}$. Now define $\tilde{\mathcal{L}}_{\mathcal{Q}}$ to be the push-out of $\pi_*\tilde{\mathcal{K}}_{\mathcal{Q}}$ by this embedding. $\tilde{\mathcal{L}}$ sits in the exact commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_*L_{\mathcal{Q}} & \longrightarrow & \pi_*\tilde{\mathcal{K}}_{\mathcal{Q}} & \longrightarrow & \mathcal{L}_{\mathcal{Q}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \tau \\ 0 & \longrightarrow & \pi_*L_{\dagger} \otimes \mathcal{Q} & \longrightarrow & \tilde{\mathcal{L}}_{\mathcal{Q}} & \longrightarrow & \Theta_S \otimes \mathcal{Q} \longrightarrow 0 \end{array}$$

Since $\pi_*L_{\dagger} \otimes \mathcal{Q}$ is contractible, we find that a left $\pi_*\tilde{\mathcal{L}}_{\mathcal{Q}}$ -module is equivalent to a left $\pi_*\mathcal{L}_{\mathcal{Q}}$ -module.

The $\pi_*\tilde{\mathcal{K}}_{\mathcal{Q}}$ -action on $R\pi_*(\mathcal{R}(\mathcal{X}/S), J(\mathcal{A}))$ induces a left $\pi_*\mathcal{L}_{\mathcal{Q}}$ -module structure on $R\pi_*(\mathcal{R}(\mathcal{X}/S), J(\mathcal{A}))$, which is the desired homotopy left action of $\pi_*\mathcal{L}$. \square

We can consider a variant of this theorem as in the previous subsection. Suppose we are given the data (a)–(e), (g) as before and

- (f') an \mathcal{O}_S -flat dg Lie algebroid \mathcal{A} on \mathcal{X}/S which is a central extension Lie $\mathcal{O}_{\mathcal{X}/S}$ -algebra,
- (h') a morphism $\bar{\tau} : L \rightarrow \mathcal{A}/\mathcal{O}_{\mathcal{X}/S}$ of Lie algebras.

These should satisfy the conditions (1)–(6) with ι replaced by $\bar{\tau}$.

Then we have an $\mathcal{O}_{\mathcal{X}/S}$ -extension L^b of L and a morphism of Lie algebras $\iota^b : L^b \rightarrow \mathcal{A}$ lifting $\bar{\tau}$. ι^b commutes with the \mathcal{K} -action on L^b which is the lift of the action (e).

Theorem 5.5.2. There is a homotopy \mathcal{O}_S -extension \mathcal{L}^b of \mathcal{L} and a left \mathcal{L}^b -module structure on $R\pi_*(\mathcal{R}(\mathcal{X}/S), J(\mathcal{A}))$.

Proof. The proof is similar with L_{\dagger} replaced by $L_{\dagger}^b := \text{Cone}(L^b \rightarrow L)$. See also the proof of Lemma 5.4.2. \square

5.6. Hitchin connection. Let us apply the discussion in the last §5.5 to the relative version of §4.3.

Let G be a semi-simple algebraic k -group and $\mathcal{C} \rightarrow S$ be a flat family of smooth k -curves of genus greater than 2. Then we have a fine moduli scheme $\mathcal{X}_s := \mathcal{M}_G(\mathcal{C}_s)$ of G -torsors on the smooth curve \mathcal{C}_s for each $s \in S$. They form a flat family $\mathcal{X} \rightarrow \mathcal{C}$ over S . Let us write $\pi : \mathcal{X} \rightarrow S$ the projection.

Set $\mathcal{A} := (\pi^{-1}\mathcal{O}_S)^G$, which is the sheaf of G -invariant vector fields on the pull-back. It is a Lie algebroid over S with the anchor $\tau : \mathcal{A} \rightarrow \Theta_S$. By §4.2 we have the short exact sequence

$$0 \longrightarrow \mathcal{A}_{\mathcal{X}/S} = \tau^{-1}\Theta_{\mathcal{X}/S} \longrightarrow \mathcal{A}_{\pi} = \tau^{-1}\Theta_{\pi} \longrightarrow \Theta_S \longrightarrow 0.$$

The Lie algebroid Θ_{π} naturally acts on \mathcal{A}_{π} . Note that the Lie algebra $\Theta_{\mathcal{X}/S}$ acts via the Θ_{π} -action.

Consider the data

- (a) The (non dg) Lie algebroid $\mathcal{L} = \Theta_S$ on S .
- (b) The Lie $\pi^{-1}\mathcal{O}_S$ -algebra $L = \Theta_{\mathcal{X}/S}$.
- (c) The Lie $\pi^{-1}\mathcal{O}_S$ -algebroid extension \mathcal{K} of $\pi^*\mathcal{L} = \Theta_{\pi}$ by $L = \Theta_{\mathcal{X}/S}$ sitting in the exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Theta_{\mathcal{X}/S} & \longrightarrow & \Theta_{\pi} & \longrightarrow & \pi^{-1}\Theta_S \longrightarrow 0 \\ & & \Delta \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{K}_0 & \xrightarrow{j} & \mathcal{K} & \longrightarrow & \pi^{-1}\mathcal{L} \longrightarrow 0 \end{array}$$

Here $\Delta : \Theta_{\mathcal{X}/S} \hookrightarrow \mathcal{K}_0 := \Theta_{\mathcal{X}/S} \otimes L = \Theta_{\mathcal{X}/S}^{\otimes 2}$ is the diagonal embedding

- (d) The section $s : \Theta_{\mathcal{X}/S} \rightarrow \mathcal{K}$ given by the first component of the morphism j in (c).
- (e) The left \mathcal{K} -module structure on $L = \Theta_{\mathcal{X}/S}$ naturally arising from $\Theta_{\mathcal{X}/S} \subset \mathcal{K}$ and the adjoint action,
- (f') The \mathcal{O}_S -flat Lie algebroid \mathcal{A}_{π} on \mathcal{X}/S .

(g) The left \mathcal{K} -module structure on \mathcal{A}_π arising from the Θ_π -action.

(h') The morphism $\bar{\tau} : L = \Theta_{\mathcal{X}/S} \rightarrow \mathcal{A}_\pi/\mathcal{O}_{\mathcal{X}/S}$ of Lie algebras on \mathcal{X}/S arising the action of $\Theta_{\mathcal{X}/S} \subset \Theta_\pi$ on \mathcal{A}_π .

The conditions (1)–(6) hold, so there is a left Θ_S^b -module structure on $R^i\pi_*(\mathcal{R}(\mathcal{X}/S), J(\mathcal{A}_\pi))$ for each i . Setting $i = 0$ and noting that a Θ_S -module is equivalent to a projective $\mathcal{D}iff_S$ -module structure by Example 4.1.1. Then seen at the second order structure, there is a projective flat connection on the space of relative global section. The first order part of $\pi_*(\mathcal{R}(\mathcal{X}/S), J(\mathcal{A}_\pi))$ is the vector bundle on S whose fiber over $s \in S$ is the space $\Gamma(\mathcal{X}_s, \det)$ of global sections of the determinant line bundle \det on the fine moduli scheme \mathcal{X}_s of G -torsors. In other words, $\Gamma(\mathcal{X}_s, \det)$ is the space of generalized theta functions. Consequently we recover Hitchin's result [H90, R06].

Theorem 5.6.1. There is a projective flat connection on the vector bundle on S with fiber the space of generalized theta functions.

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